MULTIPLETS OF REPRESENTATIONS AND KOSTANT'S DIRAC OPERATOR FOR EQUAL RANK LOOP GROUPS

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Abstract. Let $\mathfrak{g}$ be a semi-simple Lie algebra and let $\mathfrak{h}$ be a reductive subalgebra of maximal rank in $\mathfrak{g}$. Given any irreducible representation of $\mathfrak{g}$, consider its tensor product with the spin representation associated to the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. Gross, Kostant, Ramond, and Sternberg recently proved a generalization of the Weyl character formula which decomposes the signed character of this product representation in terms of the characters of a set of irreducible representations of $\mathfrak{h}$, called a multiplet. Kostant then constructed a formal $\mathfrak{h}$-equivariant Dirac operator on such product representations whose kernel is precisely the multiplet of $\mathfrak{h}$-representations corresponding to the given representation of $\mathfrak{g}$.

We reproduce these results in the Kac-Moody setting for the extended loop algebras $\tilde{L}\mathfrak{g}$ and $\tilde{L}\mathfrak{h}$. We prove a homogeneous generalization of the Weyl-Kac character formula, which now yields a multiplet of irreducible positive energy representations of $L\mathfrak{h}$ associated to any irreducible positive energy representation of $L\mathfrak{g}$. We construct a $L\mathfrak{h}$-equivariant operator, analogous to Kostant's Dirac operator, on the tensor product of a representation of $L\mathfrak{g}$ with the spin representation associated to the complement of $L\mathfrak{h}$ in $L\mathfrak{g}$. We then prove that the kernel of this operator gives the $L\mathfrak{h}$-multiplet corresponding to the original representation of $L\mathfrak{g}$.

Contents

Date: October 31, 2000.
2000 *Mathematics Subject Classification.* Primary: 17B67; Secondary: 17B35, 22E46, 81R10.

0. Introduction

Although this paper is chiefly concerned with representations of Lie groups and loop groups, the motivation for these results originally comes from M-Theory. In physics, the Lie group Spin(9) arises as the little group for massive particles in 10 dimensional superstring theories and as the little group for massless particles in 11 dimensional supergravity. Recently, Pengpan and Ramond noticed that the irreducible representations of Spin(9) come in triples, with the Casimir operator taking the same value on all three representations, and where the dimensions of two such representations sum to the dimension of the third. Ramond brought this curious fact to the attention of Sternberg, who in collaboration with Gross and Kostant then showed that these triples of representations of $B_4 = \mathrm{Spin}(9)$ actually correspond to representations of the exceptional Lie group F_4, which contains B_4 as an equal rank subgroup.

In fact, this is not an isolated phenomenon. In [2], Gross, Kostant, Ramond, and Sternberg consider the general case where $\mathfrak{h}$ is a reductive Lie algebra which is a maximal rank subalgebra of some semi-simple Lie algebra $\mathfrak{g}$. Letting G and H denote the compact, simply-connected Lie groups with Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ respectively, associated to any irreducible representation of G is a set of $\chi(G/H)$ irreducible representations of H, where $\chi(G/H)$ is the Euler number of the homogeneous space G/H. We shall refer to such a set of H-representations as a *multiplet.* As in the case of $B_4 \subset F_4$, all of the representations in a multiplet share the same value of the Casimir operator, and the alternating sum of the dimensions of these representations vanishes. The relation between a representation of G and the H-representations in the corresponding multiplet is given by the following homogeneous generalization of the Weyl character formula, viewed as an identity in the representation ring $R(H)$:

$$V_\lambda \otimes \mathbb{S}^+ - V_\lambda \otimes \mathbb{S}^- = \sum\nolimits_{c \in C} (-1)^c \, U_{c(\lambda+\rho_G)-\rho_H}, \tag{1}$$

where V_λ and U_μ denote the irreducible representations of G and H with highest weight λ and μ respectively, $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ is the spin representation associated to the complement of $\mathfrak{h}$ in $\mathfrak{g}$, the subset $C \subset W_G$ of the Weyl group of G has one representative from each coset of W_H, and $(-1)^c$ is the sign of the element c.

In representation theory, the Casimir operator of a Lie algebra is analogous to the Laplacian. Using the spin representation, we can also consider operators analogous to the Dirac operator. Furthermore, we can choose a particular Dirac operator such that its square is the Casimir operator shifted by a constant, giving a representation theory version of the Weitzenböck formula. Such a Dirac operator was introduced in a more formal setting by Alekseev and Meinrenken in [1], and the geometric version of this Dirac operator is examined in [12]. Since the Casimir operator takes the same value on all of the representations in a multiplet, it follows that this Dirac operator likewise takes a constant value, up to sign, on each multiplet.

In the homogeneous case, for any linear operator

$$\partial\!\!\!/ : V_\lambda \otimes \mathbb{S}^+ \to V_\lambda \otimes \mathbb{S}^-,$$

since both the domain and range are finite dimensional, the index of $\partial\!\!\!/$ must be given by (1). This prompted Kostant to seach for a Dirac operator whose kernel and cokernel are precisely those representations on the right hand side of (1). In [4], Kostant constructs a Dirac operator $\partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}}$ on $V_\lambda \otimes \mathbb{S}$ with a cubic term associated

to the fundamental 3-form on $\mathfrak{g}$. The kernel of Kostant's Dirac operator is

$$\operatorname{Ker} \partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}} = \bigoplus_{c\in C} U_{c(\lambda+\rho_G)-\rho_H}, \tag{2}$$

and the signs $(-1)^c$ on the right side of (1) can be recovered by decomposing the operator $\partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}}$ according to the positive and negative half-spin representations. Taking the kernel of Kostant's Dirac operator therefore gives an explicit construction of the multiplet of H-representations corresponding to a given representation of G.

This paper takes the results discussed above and reformulates them in the Kac-Moody setting, replacing the equal rank Lie groups $H \subset G$ with their corresponding loop groups $LH \subset LG$. After briefly reviewing the representation theory of loop groups in §1, we introduce the positive energy spin representation $\mathcal{S}_{L\mathfrak{g}}$ associated to a loop group in §2, using it to reformulate the Weyl-Kac character formula. In §3, we prove the following homogeneous version of the Weyl-Kac character formula:

$$\mathcal{H}_{\boldsymbol{\lambda}} \otimes \mathcal{S}^+_{L\mathfrak{g}/L\mathfrak{h}} - \mathcal{H}_{\boldsymbol{\lambda}} \otimes \mathcal{S}^-_{L\mathfrak{g}/L\mathfrak{h}} = \sum_{c\in\mathcal{C}} (-1)^c \, \mathcal{U}_{c(\boldsymbol{\lambda}-\boldsymbol{\rho}_{\mathfrak{g}})+\boldsymbol{\rho}_{\mathfrak{h}}},$$

where $\mathcal{H}_{\boldsymbol{\lambda}}$ and $\mathcal{U}_{\boldsymbol{\mu}}$ denote the positive energy representations of the central extensions $\tilde{L}G$ and $\tilde{L}H$ with lowest weights $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ respectively, the subset $\mathcal{C} \subset \mathcal{W}_G$ now lives in the affine Weyl group of G, and $-\boldsymbol{\rho}_{\mathfrak{g}}$ and $-\boldsymbol{\rho}_{\mathfrak{h}}$ are the lowest weights of the spin representations $\mathcal{S}_{L\mathfrak{g}}$ and $\mathcal{S}_{L\mathfrak{h}}$.

In §§4–6, we return to the case of compact Lie groups, reviewing various results of [1] and [4]. There we construct Kostant's Dirac operator, compute its square, and prove that its kernel has the form given by (2). Our approach here differs slightly from Kostant's, which views the Lie algebra $\mathfrak{g}$ as an orthogonal extension of $\mathfrak{h}$. Instead, we first consider the Dirac operator on $\mathfrak{g}$ and a twisted Dirac operator on $\mathfrak{h}$ and then construct Kostant's Dirac operator as their difference, an idea borrowed from [6, 7]. In addition, we avoid working with a basis for $\mathfrak{g}$ wherever possible, which greatly simplifies the computations and hopefully elucidates their meanings. These sections can stand alone as an alternative exposition on Kostant's Dirac operator, and they provide a outline of the more advanced material in the subsequent sections.

The remaining sections reprise these results for the loop group case. In §7 we examine the Clifford algebra associated to a loop group, which builds on the treatment of infinite dimensional Clifford algebras given in [5]. We then introduce the Dirac and Casimir operators associated to a loop group in §8, and we construct the loop group analogue $\partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}}$ of Kostant's Dirac operator in §9. These Dirac and Casimir operators appear in the physics literature in [8] and [6, 7] as the odd and even zero-mode generators for the $N = 1$ superconformal algebras associated to current (Lie group) and coset space (homogeneous space) models. In contrast, our treatment builds these operators on a mathematical foundation, viewing them as canonical objects rather than working in terms of a basis. Finally, we compute the square of the Dirac operator $\partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}}$, and in §10 we prove that its kernel is

$$\operatorname{Ker} \partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}} = \bigoplus_{c\in\mathcal{C}} \mathcal{U}_{c(\boldsymbol{\lambda}-\boldsymbol{\rho}_{\mathfrak{g}})+\boldsymbol{\rho}_{\mathfrak{h}}},$$

just as for compact Lie groups. So once again, taking the kernel of this Dirac operator provides an explicit construction for the multiplet of representations of $\tilde{L}H$ corresponding to any given representation of $\tilde{L}G$.

Note. Anthony Wassermann, who has independently obtained results similar to those in this paper, pointed out to me that with only minor modifications, the arguments presented here provide a quick proof of the Weyl-Kac character formula.

1. Loop groups and their representations

1.1. **Loop Groups.** Let G be a compact connected Lie group, and let LG denote the group of free loops on G, i.e., the space of smooth maps from S^1 to G, where the product of two loops is taken pointwise. The Lie algebra of the loop group LG is simply the vector space $L\mathfrak{g}$ of loops on the Lie algebra $\mathfrak{g}$ of G, with brackets again taken pointwise. The group $\mathrm{Diff}(S^1)$ of diffeomorphisms of the circle acts on loop spaces by reparameterizing the loops, and in particular the subgroup S^1 of rigid rotations of the circle acts on LG and $L\mathfrak{g}$. This circle action induces a $\mathbb{Z}$-grading on the complexified Lie algebra $L\mathfrak{g}_{\mathbb{C}}$, which is the closure of the direct sum of the Fourier components $\bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_{\mathbb{C}} z^k$, where $\mathfrak{g}_{\mathbb{C}} z^k$ denotes loops of the form $z \mapsto X z^k$ for $X \in \mathfrak{g}_{\mathbb{C}}$. We are interested in those representations of LG which likewise admit a $\mathbb{Z}$-grading intertwining with the S^1-action on LG, or in other words representations of the semi-direct product $S^1 \ltimes LG$. Such a representation $\mathcal{E}$ then decomposes into eigenspaces $\bigoplus_{k\in\mathbb{Z}} \mathcal{E}(k)$ according to the S^1-weight k, called the *energy*. (This terminology comes from an analogy with quantum mechanics, where the energies are eigenvalues of the Hamiltonian operator, which generates time translation.)

1.2. **The central extension.** The representations that we will consider are actually projective representations of LG. To realize them as true representations, we must introduce a central extension $\widetilde{L}G$ of LG by S^1. This is analogous to taking the universal cover of a compact Lie group, except that here we lift to a circle bundle rather than a finite cover.

The corresponding central extension of the Lie algebra, which is called a *Kac-Moody algebra*, is $\widetilde{L}\mathfrak{g} = L\mathfrak{g} \oplus \mathbb{R}I$, where I is the infinitesimal generator of the central S^1 subgroup. The Lie bracket on the central extension $\widetilde{L}\mathfrak{g}$ is determined by a choice of ad-invariant inner product on $L\mathfrak{g}$. Any ad-invariant inner product on the Lie algebra $\mathfrak{g}$ induces an inner product on the $L\mathfrak{g}$ by averaging the pointwise inner products. For loops $\xi, \eta \in L\mathfrak{g}$, this gives

$$\langle \xi, \eta \rangle = \frac{1}{2\pi} \int_0^{2\pi} \big\langle \xi(\theta), \eta(\theta) \big\rangle \, d\theta, \tag{3}$$

which is ad-invariant on $L\mathfrak{g}$. To extend this inner product to $\widetilde{L}\mathfrak{g}$, we must actually go one step further and extend it to the semi-direct sum $\mathbb{R} \,\tilde{\oplus}\, \widetilde{L}\mathfrak{g}$, where $\mathbb{R}$ is generated by the infinitesimal rotation ∂_θ, and we define the inner product by

$$\langle a\,\partial_\theta + \xi + xI,\, b\,\partial_\theta + \eta + yI \rangle = \langle \xi, \eta \rangle - ay - bx$$

for $a, b, x, y \in \mathbb{R}$ and $\xi, \eta \in L\mathfrak{g}$. This inner product is ad-invariant on the extended Lie algebra $\mathbb{R} \,\tilde{\oplus}\, \widetilde{L}\mathfrak{g}$ provided that the Lie bracket on the central extension $\widetilde{L}\mathfrak{g}$ is

$$[\xi, \eta]_{\widetilde{L}\mathfrak{g}} = [\xi, \eta]_{L\mathfrak{g}} + \langle \xi, \partial_\theta \eta \rangle \, I. \tag{4}$$

Although this central extension depends on the original choice of inner product on $\mathfrak{g}$, there is a unique ad-invariant inner product on $\mathfrak{g}$ (up to scaling) if $\mathfrak{g}$ is *simple*. In this case, the *universal central extension* corresponds to the smallest possible scaling for which the Lie algebra $\widetilde{L}\mathfrak{g}$ exponentiates to give a central extension $\widetilde{L}G$ of the loop group LG. This smallest inner product on $\mathfrak{g}$ is the *basic inner product*, which is scaled so that the highest root $\alpha_{\max}$ of $\mathfrak{g}$ satisfies $\|\alpha_{\max}\|^2 = 2$.

If G is not simple but only semi-simple, then a given projective representation of LG can still be lifted to a true representation of some S^1 extension of LG. However, the universal central extension of LG is no longer a circle bundle, but rather an

extension by the torus T^d, where d counts the number of simple components of G. At the Lie algebra level, an ad-invariant inner product on $\mathfrak{g}$ can be scaled separately on each of the simple components, and the central term in the Lie bracket (4) now becomes d separate terms corresponding to the basic inner products for each of these components.

Remark. Let G be simply connected. Topologically, the invariant inner products on $\mathfrak{g}$ correspond to elements of the Lie algebra cohomology $H^3(\mathfrak{g}) \cong H^3(G;\mathbb{R})$ by associating to any inner product its fundamental 3-form $\omega \in \Lambda^3(\mathfrak{g}^*)$ given by $\omega(X,Y,Z) = \langle X,[Y,Z]\rangle$ for $X,Y,Z \in \mathfrak{g}$. The possible central extensions of the Lie algebra $L\mathfrak{g}$ by a circle thus correspond to elements of the real cohomology $H^3(G;\mathbb{R})$, and the universal central extension of $L\mathfrak{g}$ is then an extension by the dual space $K = H_3(G;\mathbb{R})$. On the other hand, the central extensions of the loop group LG correspond to circle bundles, which are classified by their Chern classes $c_1 \in H^2(LG;\mathbb{Z}) \cong H^3(G;\mathbb{Z})$ in the integral lattice of $H^3(G;\mathbb{Z})$. Writing $L = H_3(G;\mathbb{Z})$ for the dual lattice in K, the universal central extension $\tilde{L}G$ is an extension of LG by the torus K/L. Using the cohomology spectral sequence for this extension and noting that $H^1(LG;\mathbb{Z}) = H^2(G;\mathbb{Z}) = 0$, we obtain the exact sequence

$$0 \to H^1(\tilde{L}G;\mathbb{Z}) \to H^1(K/L;\mathbb{Z}) \xrightarrow{d_2} H^2(LG;\mathbb{Z}) \to H^2(\tilde{L}G;\mathbb{Z}) \to 0.$$

Now, by our construction of the torus K/L, we have a canonical isomorphism $H^1(K/L;\mathbb{Z}) \cong H^3(G;\mathbb{Z})$, and we also have a canonical isomorphism $H^2(LG;\mathbb{Z}) \cong H^3(G;\mathbb{Z})$. The map d_2 is therefore a homomorphism $d_2 : H^3(G;\mathbb{Z}) \to H^3(G;\mathbb{Z})$, and the universality condition becomes the assertion that d_2 be the identity map. In particular, if $\tilde{L}G$ is the universal central extension, then d_2 must be an isomorphism, and it follows that $H^1(\tilde{L}G;\mathbb{Z}) = H^2(\tilde{L}G;\mathbb{Z}) = 0$, which in terms of homotopy implies that $\tilde{L}G$ is 2-connected. So, whereas taking the universal cover of a compact Lie group G kills the obstruction $\pi_1(G)$, the loop group LG is already simply connected, but taking its universal central extension kills the obstruction $\pi_2(LG)$.

The semi-direct sum $\mathbb{R} \tilde{\oplus} \tilde{L}\mathfrak{g}$ which we introduced above is the Lie algebra of the semi-direct product $S^1 \ltimes \tilde{L}G$, and from here on we refer to representations of $S^1 \ltimes \tilde{L}G$ as representations of LG. Given such a representation, we call the weight of the central S^1 in $\tilde{L}G$ the *level* or *central charge*, and since this circle by definition commutes with the rest of the loop group, it follows that the level is constant on each irreducible representation. Unless stated otherwise, from here on we assume that G is simply connected and simple, we use the basic inner product on $\mathfrak{g}$, and we let $\tilde{L}G$ denote the universal central extension. However, the following discussion can be generalized to the semi-simple case by treating the d simple components separately and viewing the level as a d-vector.

1.3. **Affine roots and the affine Weyl group.** Let T be a maximal torus of G. When considering the representation theory of loop groups, rather than taking the abelian subgroup LT as the maximal torus of LG, we instead use the maximal torus $S^1 \times T \times S^1$ of $S^1 \ltimes \tilde{L}G$. Here the first S^1 factor corresponds to rotation of loops, while the second comes from the central extension. The Cartan subalgebra is then $\mathbb{R} \oplus \mathfrak{t} \oplus \mathbb{R}$, and the weights of LG are of the form $\boldsymbol{\lambda} = (m,\lambda,h)$, where m is the energy, λ is a weight of G, and h is the level. In this notation, the roots of LG, also called the *affine roots* of G, consist of the weights $(m,\alpha,0)$ with $m \in \mathbb{Z}$ and α a root of G, as well as the weights $(m,0,0)$ for nonzero m, counted with

multiplicity $\operatorname{rank} G = \dim \mathfrak{t}$. Given a system of positive roots for G, we take the positive roots of LG to be the roots $(0, \alpha, 0)$ for $\alpha > 0$, as well as all roots $(m, \alpha, 0)$ with $m > 0$, including roots of the form $(m, 0, 0)$. If $\{\alpha_i\}$ is a set of simple roots for G, then the corresponding simple affine roots for LG are $(0, \alpha_i, 0)$, as well as the root $(1, -\alpha_{\max}, 0)$, where $\alpha_{\max}$ is the highest root of G.

The affine Weyl group $\mathcal{W}_G$ of G is the group generated by the reflections through the hyperplanes corresponding to the affine roots of G. In terms of loop groups, given any root $\boldsymbol{\alpha} = (k, \alpha, 0)$ of LG, there is a corresponding $\mathfrak{su}(2)$ subalgebra of $\tilde{L}\mathfrak{g}$ generated by the loops $E_\alpha z^k$ and $E_{-\alpha} z^{-k}$ and the coroot

$$H_{k,\alpha} = \left[E_\alpha z^k, E_{-\alpha} z^{-k}\right]_{\tilde{L}\mathfrak{g}} = H_\alpha + \tfrac{1}{2} i k \|H_\alpha\|^2 I,$$

where $\{E_\alpha, E_{-\alpha}, H_\alpha\}$ span the $\mathfrak{su}(2)$ subalgebra of $\mathfrak{g}$ associated to the root α. Note that these elements are normalized so that $\langle E_\alpha, E_{-\alpha}\rangle = \frac{1}{2}\|H_\alpha\|^2 = 2\langle \alpha, \alpha\rangle^{-1}$. The reflection of a weight $\boldsymbol{\lambda} = (m, \lambda, h)$ through the hyperplane orthogonal to $\boldsymbol{\alpha}$ is then

$$\begin{aligned} s_{k,\alpha}(\boldsymbol{\lambda}) &= \boldsymbol{\lambda} - \boldsymbol{\lambda}(H_{k,\alpha})\boldsymbol{\alpha} \\ &= \left(m - \lambda(H_\alpha)k + \tfrac{1}{2} h \|H_\alpha\|^2 k^2, \lambda - \lambda(H_\alpha)\alpha + \tfrac{1}{2} h \|H_\alpha\|^2 k\alpha, h\right). \end{aligned} \tag{5}$$

Furthermore, these $s_{k,\alpha}$ are generated by the reflections $s_{0,\alpha}$, which act solely on the $\mathfrak{t}^*$ component and generate the usual Weyl group W_G, as well as the transformations

$$t_\alpha(\boldsymbol{\lambda}) = s_{1,\alpha} s_{0,\alpha}(\boldsymbol{\lambda}) = \left(m + \lambda(H_\alpha) + \tfrac{1}{2} h \|H_\alpha\|^2, \lambda + hH_\alpha, h\right),$$

where we use the inner product to identify the coroot $H_\alpha \in \mathfrak{t}$ with the weight $\frac{1}{2}\|H_\alpha\|^2 \alpha$ in $\mathfrak{t}^*$. Restricting to $\mathfrak{t}^*$, the t_α are simply translations by the coroots, which generate the coweight lattice $L \subset \mathfrak{t}$. We therefore have $\mathcal{W}_G \cong W_G \ltimes L$.

Note that under the action of the affine Weyl group, the level h is fixed, while the energy m is shifted so as to preserve the inner product

$$(m_1, \lambda_1, h_1) \cdot (m_2, \lambda_2, h_2) = \langle \lambda_1, \lambda_2\rangle - m_1 h_2 - m_2 h_1 \tag{6}$$

on $\mathbb{R} \oplus \mathfrak{t}^* \oplus \mathbb{R}$. Thus, at any given level h, the affine Weyl action is completely determined by its restriction to $\mathfrak{t}^*$. In particular, the element $s_{k,\alpha}$ corresponds to the reflection through the hyperplane given by the equation $\langle \lambda, \alpha\rangle = hk$. These hyperplanes divide $\mathfrak{t}^*$ into connected components called *alcoves*, and the affine Weyl group acts simply transitively on these alcoves. Given a positive root system for LG, the corresponding *fundamental alcove* is the unique alcove satisfying $\boldsymbol{\lambda} \cdot \boldsymbol{\alpha} \leq 0$ for all $\boldsymbol{\alpha} > 0$. This alcove is bounded by the hyperplanes corresponding to the negatives of the simple affine roots, or in other words, a weight $\boldsymbol{\lambda} = (m, \lambda, h)$ lies in the fundamental alcove if and only if $-\lambda$ is in the positive Weyl chamber for G and $\langle \lambda, -\alpha_{\max}\rangle \leq h$.

1.4. **Positive energy representations.** A representation $\mathcal{H}$ of LG is a *positive energy representation* if $\mathcal{H}(k) = 0$ for all $k < m$ for some fixed integer m, or in other words, there is a minimum energy when $\mathcal{H}$ is decomposed into its constant energy eigenspaces. In the literature, positive energy representations are often normalized so that this minimum energy is 0. However, we will consider positive energy representations with the full spectrum of minimum energies. When restricted to the positive energy representations, the representation theory of loop groups behaves quite analogously to the representation theory of compact Lie groups. In particular, the positive energy representations satisfy the following fundamental properties (for a complete discussion, see [11]):

(i) A positive energy representation is *completely reducible* into a direct sum of (possibly infinitely many) irreducible positive energy representations.
(ii) An irreducible positive energy representation $\mathcal{H}$ is of *finite type*: each of the constant energy subspaces $\mathcal{H}(k)$ is a finite dimensional representation of G.
(iii) Every irreducible positive energy representation $\mathcal{H}$ has a unique *lowest weight* $\boldsymbol{\lambda} = (m, \lambda, h)$, in the sense that $\boldsymbol{\lambda} - \boldsymbol{\alpha}$ is not a weight of $\mathcal{H}$ for any positive root $\boldsymbol{\alpha}$ of LG. The lowest weight space is one dimensional and generates $\mathcal{H}$.
(iv) A weight $\boldsymbol{\lambda} = (m, \lambda, h)$ is *anti-dominant* for LG if it lies in the fundamental Weyl alcove described at the end of §1.3 above. The lowest weight of a positive energy representation is anti-dominant, and every anti-dominant weight is realized as the lowest weight of some positive energy representation.

As a consequence of (iii), an irreducible positive energy representation $\mathcal{H}$ is completely characterized by its minimum energy m, its minimum energy subspace $\mathcal{H}(m) \cong V_{-\lambda}$, and its level h. Property (iv) implies that for a positive energy representation, the level h is always non-negative and is zero only for the trivial representation. Also, for a fixed minimum energy m, there are only finitely many positive energy representations at each level h, but as the level tends to infinity, the representation theory of LG resembles that of G.

If $\mathcal{H}_{\boldsymbol{\lambda}}$ is the irreducible positive energy representation with lowest weight $\boldsymbol{\lambda} = (0, \lambda, h)$, then $\mathcal{H}_{\boldsymbol{\lambda}}$ also contains all the weights in the orbit of $\boldsymbol{\lambda}$ under the affine Weyl group $\mathcal{W}_G$. Recalling that the affine Weyl group action preserves the inner product (6), it turns out that the orbit of $\boldsymbol{\lambda}$ consists of all weights $\boldsymbol{\mu} = (m, \mu, h)$ at level h satisfying $\boldsymbol{\lambda} \cdot \boldsymbol{\lambda} = \boldsymbol{\mu} \cdot \boldsymbol{\mu}$, or equivalently $\|\mu\|^2 - 2mh = \|\lambda\|^2$. This equation sweeps out a paraboloid, and the weights of $\mathcal{H}_{\boldsymbol{\lambda}}$ all lie in its interior. (As the level h tends to infinity, this paraboloid flattens into a cone.) For an example, see Figure 1 at the end of §3, which gives the weights of the irreducible representation of $L\mathrm{SU}(2)$ with lowest weight $(0, -1, 2)$.

2. The spin representation

If V is a finite dimensional vector space with an inner product, and $V = W \oplus W^*$ is a polarization of V into a maximal isotropic subspace W and its dual, then the spin representation of the Clifford algebra $\mathrm{Cl}(V)$ can be written in the form

$$\mathbb{S}_V = \Lambda^*(W) \otimes (\det W)^{-\frac{1}{2}}, \tag{7}$$

where $\det W$ denotes the top exterior power of W. The resulting spin representation $\mathbb{S}_V$ is independent of the choice of polarization, which is accounted for by the factor of $(\det W)^{-1/2}$. On the other hand, if V is infinite dimensional, then this determinant factor does not make sense, and so we can no longer use (7) to define the spin representation. Without this determinant factor to correct for the choice of polarization, different polarizations give rise to distinct spin representations. For a general discussion of infinite dimensional Clifford algebras and their spin representations, see [5].

For our purposes, consider the Lie algebra $L\mathfrak{g}$ with the inner product (3) induced by the basic inner product on $\mathfrak{g}$. If we complexify $L\mathfrak{g}$, then the orthogonal complement of the Cartan subalgebra $\mathfrak{t}_{\mathbb{C}}$ in $L\mathfrak{g}_{\mathbb{C}}$ decomposes into the sum of the positive and negative root spaces, each of which is isotropic with respect to the inner product on $L\mathfrak{g}_{\mathbb{C}}$. We can therefore use this polarization to define a positive

energy spin representation associated to the complement of $\mathfrak{t}$ in $L\mathfrak{g}$:

$$\mathcal{S}_{L\mathfrak{g}/\mathfrak{t}} := \mathbb{S}_{\mathfrak{g}/\mathfrak{t}} \otimes \Lambda^*\Big(\bigoplus\nolimits_{k>0} \mathfrak{g}_{\mathbb{C}} z^k\Big) = \mathbb{S}_{\mathfrak{g}/\mathfrak{t}} \otimes \bigotimes\nolimits_{k>0} \Lambda^*\big(\mathfrak{g}_{\mathbb{C}} z^k\big), \tag{8}$$

where we have explicitly factored out the contribution $\mathbb{S}_{\mathfrak{g}/\mathfrak{t}}$ coming from the constant loops (or zero modes). Here, we have used the expression (7) for the spin representation, except that we have dropped the portion of the $(\det W)^{-1/2}$ factor coming from the positive energy modes. If we were to include that factor, it would contribute an overall anomalous energy shift of

$$\Big(\prod\nolimits_{k>0} z^{k \dim \mathfrak{g}}\Big)^{-\frac{1}{2}} = z^{-\frac{1}{2}\sum_{k>0} k \dim \mathfrak{g}} = z^{\frac{1}{24}\dim \mathfrak{g}}, \tag{9}$$

where in the last equality we use the Riemann zeta function trick to write the infinite sum as $\sum_{k>0} k = \zeta(-1) = -\frac{1}{12}$. Fortunately, by normalizing the spin representation to have minimum energy 0, we can safely ignore this factor.

For the moment, we are interested only in the character of the spin representation. The restriction of the character of $\mathcal{S}_{L\mathfrak{g}/\mathfrak{t}}$ to $S^1 \times T$ is completely determined by the description (8) of the spin representation. However, in correcting for the infinite determinant factor, the spin representation acquires a nonzero central charge.

Theorem 1. *If G is simple, then the central charge of the spin representation $\mathcal{S}_{L\mathfrak{g}/\mathfrak{t}}$ is the value of the quadratic Casimir operator of $\mathfrak{g}$ in the adjoint representation:*

$$c_G = \Delta^{\mathfrak{g}}_{\mathrm{ad}} = -\frac{1}{2}\sum\nolimits_i (\mathrm{ad}\, X_i)^2 = \langle \rho_G, \alpha_{\max}\rangle + 1,$$

where ρ_G is half the sum of the positive roots, $\alpha_{\max}$ is the highest root of G, and $\{X_i\}$ is an orthonormal basis for $\mathfrak{g}$.

Proof. To compute the central charge of the spin representation $\mathcal{S}_{L\mathfrak{g}/\mathfrak{t}}$, we extend it to obtain the spin representation associated to the entire Lie algebra $L\mathfrak{g}$. Since the construction of spin representations is multiplicative, we have

$$\mathcal{S}_{L\mathfrak{g}} \cong \mathbb{S}_{\mathfrak{t}} \otimes \mathcal{S}_{L\mathfrak{g}/\mathfrak{t}}.$$

These two spin representations have the same central charge since they differ only by the finite dimensional factor $\mathbb{S}_{\mathfrak{t}}$. However, the extended spin representation $\mathcal{S}_{L\mathfrak{g}}$ admits an action of the full Lie algebra $\tilde{L}\mathfrak{g}$, and in fact, $\mathcal{S}_{L\mathfrak{g}}$ is the direct sum of $\dim \mathbb{S}_{\mathfrak{t}}$ copies of an irreducible positive energy representation of $L\mathfrak{g}$. Examining the structure of this representation, the first three energy levels of $\mathcal{S}_{L\mathfrak{g}}$ are as follows:

$$\begin{aligned}
\mathcal{S}_{L\mathfrak{g}}(0) &= \mathbb{S}_{\mathfrak{g}},\\
\mathcal{S}_{L\mathfrak{g}}(1) &= \mathbb{S}_{\mathfrak{g}} \otimes \mathfrak{g}_{\mathbb{C}},\\
\mathcal{S}_{L\mathfrak{g}}(2) &= \mathbb{S}_{\mathfrak{g}} \otimes \mathfrak{g}_{\mathbb{C}} \,\oplus\, \mathbb{S}_{\mathfrak{g}} \otimes \Lambda^2(\mathfrak{g}_{\mathbb{C}}).
\end{aligned}$$

Letting α denote the highest root of $\mathfrak{g}$, and c the central charge of $\mathcal{S}_{L\mathfrak{g}}$, the highest weights of $\mathcal{S}_{L\mathfrak{g}}(0)$ and $\mathcal{S}_{L\mathfrak{g}}(1)$ are then $(0, \rho, c)$ and $(1, \rho+\alpha, c)$ respectively, while the weight $(2, \rho+2\alpha, c)$ is *not* present in $\mathcal{S}_{L\mathfrak{g}}(2)$. The weights $(0, \rho, c)$ and $(1, \rho+\alpha, c)$ thus form a complete string of weights for the root $\boldsymbol{\alpha} = (1, \alpha, 0)$, and so they must be related to each other by the affine Weyl element $s_{1,\alpha}$, the reflection through the hyperplane orthogonal to $\boldsymbol{\alpha}$. By (5), the difference of these weights is $(1, \rho+\alpha, c) - (0, \rho, c) = \boldsymbol{\alpha} = -(0, \rho, c)(H_{1,\alpha})\boldsymbol{\alpha}$, so we obtain

$$-1 = (0, \rho, c)(H_{1,\alpha}) = \rho(H_\alpha) - \tfrac{1}{2}\|H_\alpha\|^2 c = \langle \rho, \alpha\rangle - c,$$

where $\frac{1}{2}\|H_\alpha\|^2 = 1$ and $\rho(H_\alpha) = \langle \rho, \alpha \rangle$ in the basic inner product since α is the highest root. The central charge of the spin representation is thus $c = \langle \rho, \alpha \rangle + 1$.

The quadratic Casimir operator of a Lie algebra does not depend on the choice of orthonormal basis, and it commutes with the action of the Lie algebra. It therefore acts by a constant times the identity on each irreducible representation. On the irreducible representation of highest weight α, the value of the Casimir operator is $\frac{1}{2}\|\alpha\|^2 + \langle \alpha, \rho \rangle$. In particular, if G is simple, then the adjoint representation is irreducible, and taking α to be the highest root of G, which satisfies $\|\alpha\|^2 = 2$ in the basic inner product, we again obtain the value $\langle \rho, \alpha \rangle + 1$ as desired. □

We can now compute the character of $\mathcal{S}_{L\mathfrak{g}/\mathfrak{t}}$ directly from the decomposition (8) and Theorem 1. Written in terms of the affine roots $\boldsymbol{\alpha} = (k, \alpha, 0)$, the character is

$$\chi(\mathcal{S}_{L\mathfrak{g}/\mathfrak{t}}) = u^{c_G} \prod_{\alpha > 0} \left(e^{\frac{i\alpha}{2}} + e^{-\frac{i\alpha}{2}}\right) \prod_{k>0,\, \alpha} \left(1 + e^{i\alpha} z^k\right) = e^{-i\boldsymbol{\rho}_G} \prod_{\boldsymbol{\alpha} > 0} (1 + e^{i\boldsymbol{\alpha}}),$$

where u is a parameter on the central S^1 extension in $\tilde{L}G$, and $\boldsymbol{\rho}_G = (0, \rho_G, -c_G)$. Here, $-\boldsymbol{\rho}_G$ is the lowest weight of $\mathcal{S}_{L\mathfrak{g}/\mathfrak{t}}$, which corresponds to the square root of the determinant in (7). This weight is the loop group version of ρ_G, half the sum of the positive roots of G, which is also characterized by the identity $\rho_G(H_\alpha) = 1$ for each of the simple roots α of G. In the loop group case, the identity $\boldsymbol{\rho}_G(H_{\boldsymbol{\alpha}}) = 1$ must hold as $\boldsymbol{\alpha}$ ranges over the simple *affine* roots, including the additional root $(1, -\alpha_{\max}, 0)$. However, in our proof of Theorem 1, the condition $\boldsymbol{\rho}_G(H_{1,-\alpha_{\max}}) = 1$ is the same equation (up to sign) that we used to compute the central charge c_G.

The spin representation decomposes as $\mathcal{S}_{L\mathfrak{g}/\mathfrak{t}} = \mathcal{S}^+_{L\mathfrak{g}/\mathfrak{t}} \oplus \mathcal{S}^-_{L\mathfrak{g}/\mathfrak{t}}$ into the sum of two half-spin representations. In particular, since the complement of $\mathfrak{t}$ in $\mathfrak{g}$ is even dimensional, the zero mode factor $\mathbb{S}_{\mathfrak{g}/\mathfrak{t}}$ of $\mathcal{S}_{L\mathfrak{g}/\mathfrak{t}}$ splits into half-spin representations, and the exterior algebra in (8) splits into its even and odd degree components. The difference of the characters of these half-spin representations is

$$\chi\left(\mathcal{S}^+_{L\mathfrak{g}/\mathfrak{t}}\right) - \chi\left(\mathcal{S}^-_{L\mathfrak{g}/\mathfrak{t}}\right) = e^{-i\boldsymbol{\rho}_G} \prod\nolimits_{\boldsymbol{\alpha} > 0} \left(1 - e^{i\boldsymbol{\alpha}}\right), \tag{10}$$

which can be viewed either as a supertrace on $\mathcal{S}_{L\mathfrak{g}/\mathfrak{t}}$ or as the character of the virtual representation $\mathcal{S}^+_{L\mathfrak{g}/\mathfrak{t}} - \mathcal{S}^-_{L\mathfrak{g}/\mathfrak{t}}$. Using the notation of spin representations, the Weyl-Kac character formula becomes

Theorem 2 (Weyl-Kac Character Formula)**.** *If G is simply connected and simple, then the character of the irreducible positive energy representation $\mathcal{H}_{\boldsymbol{\lambda}}$ of $\tilde{L}G$ with lowest weight $\boldsymbol{\lambda}$ is given by the quotient*

$$\chi(\mathcal{H}_{\boldsymbol{\lambda}}) = \frac{\sum_{w \in \mathcal{W}_G} (-1)^w e^{iw(\boldsymbol{\lambda} - \boldsymbol{\rho}_G)}}{\chi\left(\mathcal{S}^+_{L\mathfrak{g}/\mathfrak{t}}\right) - \chi\left(\mathcal{S}^-_{L\mathfrak{g}/\mathfrak{t}}\right)}, \tag{11}$$

where $\mathcal{W}_G$ is the affine Weyl group of G and $\boldsymbol{\rho}_G = (0, \rho_G, -c_G)$.

Note that as an immediate consequence of the Weyl-Kac character formula, if we consider the trivial representation with $\boldsymbol{\lambda} = 0$, we obtain the identity

$$\chi\left(\mathcal{S}^+_{L\mathfrak{g}/\mathfrak{t}}\right) - \chi\left(\mathcal{S}^-_{L\mathfrak{g}/\mathfrak{t}}\right) = \sum\nolimits_{w \in \mathcal{W}_G} (-1)^w e^{-iw(\boldsymbol{\rho}_G)},$$

which gives an alternative expression for the signed character (10) of the spin representation appearing in the denominator of (11).

Remark. If G is semi-simple, then we recall that the universal central extension of LG is an extension not by a circle but rather by the torus T^d, where d counts the number of simple components. In this case the central charge of the spin representation is the d-vector $\mathbf{c}_G = (c_{G_1}, \ldots, c_{G_d})$, where $G_1, \ldots, G_d$ are the simple components of G. If we work with the universal central extension of LG and define $\boldsymbol{\rho}_G = (0, \rho_G, -\mathbf{c}_G)$, then the Weyl-Kac character formula still holds as written. In fact, using the appropriate universal central extension, the Weyl-Kac character formula continues to hold for an arbitrary compact Lie group G.

3. The homogeneous Weyl-Kac formula

Let $\mathfrak{g}$ be a compact, semi-simple Lie algebra, and let $\mathfrak{h}$ be a reductive subalgebra of maximal rank in $\mathfrak{g}$. In [2], Gross, Kostant, Ramond, and Sternberg prove a homogeneous generalization of the Weyl character formula, associating to each $\mathfrak{g}$-representation a set of $\mathfrak{h}$-representations with similar properties, called a multiplet.

Theorem 3 (Homogeneous Weyl Formula)**.** *Let V_λ and U_μ denote the irreducible representations of $\mathfrak{g}$ and $\mathfrak{h}$ with highest weights λ and μ respectively. The following identity holds in the representation ring $R(\mathfrak{h})$:*

$$V_\lambda \otimes \mathbb{S}^+_{\mathfrak{g}/\mathfrak{h}} - V_\lambda \otimes \mathbb{S}^-_{\mathfrak{g}/\mathfrak{h}} = \sum\nolimits_{c \in C} (-1)^c \, U_{c(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{h}}}, \tag{12}$$

where the sum is taken over the subset C of elements $c \in W_{\mathfrak{g}}$ of the Weyl group of $\mathfrak{g}$ for which $c(\lambda + \rho_{\mathfrak{g}}) - \rho_{\mathfrak{h}}$ are dominant weights of $\mathfrak{h}$.

Note that if $\mathfrak{h} = \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$, then C is the full Weyl group $W_{\mathfrak{g}}$, and (12) becomes the Weyl character formula. Also note that by stating this result in terms of the Lie algebras $\mathfrak{h} \subset \mathfrak{g}$ rather than their corresponding Lie groups $H \subset G$, we bypass the issue of whether the spin representation $\mathbb{S}_{\mathfrak{g}/\mathfrak{h}}$ exponentiates to give a true representation of H. Geometrically, this is equivalent to the condition that G/H be a spin manifold.

Theorem 3 has an immediate analogue for loop groups. The only complication is that simply working at the level of Lie algebras is no longer sufficient to avoid the geometric obstruction, which in this case is the condition that G/H admit a *string structure* (see [10]). Rather, we must work with the universal central extensions. Given $\mathfrak{g}$ and $\mathfrak{h}$ as described above, let $\tilde{L}\mathfrak{g}$ be the universal central extension of $L\mathfrak{g}$, and let $\tilde{L}\mathfrak{h}$ be the restriction of $\tilde{L}\mathfrak{g}$ to $L\mathfrak{h}$. Note that $\tilde{L}\mathfrak{h}$ is not in general the universal central extension of $L\mathfrak{h}$, which we denote by $\hat{L}\mathfrak{h}$. Rather, $\tilde{L}\mathfrak{h}$ is a quotient of $\hat{L}\mathfrak{h}$. Since $\mathfrak{h}$ has the same rank as $\mathfrak{g}$, if $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{h}$, then it is likewise a Cartan subalgebra of $\mathfrak{g}$. The Cartan subalgebras of $\mathbb{R} \tilde{\oplus} \hat{L}\mathfrak{h}$ and $\mathbb{R} \tilde{\oplus} \tilde{L}\mathfrak{g}$ are then $\mathbb{R} \oplus \mathfrak{t} \oplus \mathbb{R}^{d_{\mathfrak{h}}}$ and $\mathbb{R} \oplus \mathfrak{t} \oplus \mathbb{R}^{d_{\mathfrak{g}}}$ respectively, where $d_{\mathfrak{g}}$ is the number of simple components of $\mathfrak{g}$ and $d_{\mathfrak{h}} \geq d_{\mathfrak{g}}$. In other words, we have the commutative diagram

$$\begin{array}{ccccc}
\mathbb{R} \oplus \mathfrak{t} \oplus \mathbb{R}^{d_{\mathfrak{h}}} & \xrightarrow{\text{quotient}} & \mathbb{R} \oplus \mathfrak{t} \oplus \mathbb{R}^{d_{\mathfrak{g}}} & = & \mathbb{R} \oplus \mathfrak{t} \oplus \mathbb{R}^{d_{\mathfrak{g}}} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{R} \tilde{\oplus} \hat{L}\mathfrak{h} & \xrightarrow{\text{quotient}} & \mathbb{R} \tilde{\oplus} \tilde{L}\mathfrak{h} & \xrightarrow{\text{inclusion}} & \mathbb{R} \tilde{\oplus} \tilde{L}\mathfrak{g}
\end{array}$$

where the vertical arrows are inclusions of Cartan subalgebras.

The weights of $L\mathfrak{h}$ and $L\mathfrak{g}$ live in the dual spaces to their Cartan subalgebras, and dual to the quotient map we have an inclusion

$$\mathbb{R} \oplus \mathfrak{t}^* \oplus \mathbb{R}^{d_{\mathfrak{g}}} \longrightarrow \mathbb{R} \oplus \mathfrak{t}^* \oplus \mathbb{R}^{d_{\mathfrak{h}}}.$$

We may therefore view the weight lattice of $L\mathfrak{g}$ as a subset of the weight lattice of $L\mathfrak{h}$. On the other hand, if we ignore the central extension (i.e., restrict to weights of level 0), then the weight lattices are identical, and the roots of $L\mathfrak{h}$ are a subset of the roots of $L\mathfrak{g}$. Consequently, the affine Weyl group $\mathcal{W}_\mathfrak{h}$ of $\mathfrak{h}$, which is generated by the reflections through the hyperplanes orthogonal to the roots of $L\mathfrak{h}$, is a subgroup of the affine Weyl group $\mathcal{W}_\mathfrak{g}$ of $\mathfrak{g}$.

Theorem 4 (Homogeneous Weyl-Kac Formula)**.** *Let $\mathcal{H}_{\boldsymbol{\lambda}}$ and $\mathcal{U}_{\boldsymbol{\mu}}$ denote the irreducible positive energy representations of $\tilde{L}\mathfrak{g}$ and $\hat{L}\mathfrak{h}$ with lowest weights $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ respectively. We then have the following identity for virtual representations of $\hat{L}\mathfrak{h}$:*

$$\mathcal{H}_{\boldsymbol{\lambda}} \otimes \mathcal{S}^+_{L\mathfrak{g}/L\mathfrak{h}} - \mathcal{H}_{\boldsymbol{\lambda}} \otimes \mathcal{S}^-_{L\mathfrak{g}/L\mathfrak{h}} = \sum\nolimits_{c\in\mathcal{C}} (-1)^c\, \mathcal{U}_{c(\boldsymbol{\lambda}-\boldsymbol{\rho}_\mathfrak{g})+\boldsymbol{\rho}_\mathfrak{h}}\,, \tag{13}$$

where the sum is taken over the subset $\mathcal{C}$ of elements $c \in \mathcal{W}_\mathfrak{g}$ of the affine Weyl group of $\mathfrak{g}$ for which $c(\boldsymbol{\lambda}-\boldsymbol{\rho}_\mathfrak{g})+\boldsymbol{\rho}_\mathfrak{h}$ are anti-dominant weights of $\hat{L}\mathfrak{h}$.

Proof. We first note that the construction of the spin representation is multiplicative, provided that the underlying vector spaces are even dimensional. In our case, the positive and negative energy subspaces pair off, while for the zero modes, the maximal rank condition implies that the complement of $\mathfrak{h}$ in $\mathfrak{g}$ and the complement of $\mathfrak{t}$ in $\mathfrak{h}$ are even dimensional, so we have

$$\mathcal{S}^+_{L\mathfrak{g}/\mathfrak{t}} - \mathcal{S}^-_{L\mathfrak{g}/\mathfrak{t}} = \big(\mathcal{S}^+_{L\mathfrak{g}/L\mathfrak{h}} - \mathcal{S}^-_{L\mathfrak{g}/L\mathfrak{h}}\big) \otimes \big(\mathcal{S}^+_{L\mathfrak{h}/\mathfrak{t}} - \mathcal{S}^-_{L\mathfrak{h}/\mathfrak{t}}\big). \tag{14}$$

Applying the Weyl-Kac character formula (11) to the left side of (13), and factoring the Weyl-Kac denominator using (14), we obtain

$$\chi\big(\mathcal{H}_{\boldsymbol{\lambda}} \otimes \mathcal{S}^+_{L\mathfrak{g}/L\mathfrak{h}}\big) - \chi\big(\mathcal{H}_{\boldsymbol{\lambda}} \otimes \mathcal{S}^-_{L\mathfrak{g}/L\mathfrak{h}}\big) = \frac{\sum_{w\in\mathcal{W}_\mathfrak{g}} (-1)^w e^{iw(\boldsymbol{\lambda}-\boldsymbol{\rho}_\mathfrak{g})}}{\chi\big(\mathcal{S}^+_{L\mathfrak{h}/\mathfrak{t}}\big) - \chi\big(\mathcal{S}^-_{L\mathfrak{h}/\mathfrak{t}}\big)}. \tag{15}$$

Recall that the affine Weyl group acts simply transitively on the Weyl alcoves. Due to the $\boldsymbol{\rho}_\mathfrak{g}$ shift, the weight $\boldsymbol{\lambda}-\boldsymbol{\rho}_\mathfrak{g}$ lies in the interior of the fundamental Weyl alcove for $\mathfrak{g}$, and thus for any $w \in \mathcal{W}_\mathfrak{g}$, the weight $w(\boldsymbol{\lambda}-\boldsymbol{\rho}_\mathfrak{g})$ likewise lies in the interior of some Weyl alcove. Furthermore, the Weyl alcoves for $\mathfrak{g}$ are completely contained inside the Weyl alcoves for $\mathfrak{h}$, and so there exists a unique element $w' \in \mathcal{W}_\mathfrak{h}$ such that $w'w(\boldsymbol{\lambda}-\boldsymbol{\rho}_\mathfrak{g})$ lies in the interior of the fundamental Weyl alcove for $\mathfrak{h}$. Shifting back by $\boldsymbol{\rho}_\mathfrak{h}$, we see that the weight $w'w(\boldsymbol{\lambda}-\boldsymbol{\rho}_\mathfrak{g})+\boldsymbol{\rho}_\mathfrak{h}$ is anti-dominant for $\hat{L}\mathfrak{h}$. Putting $c = w'w$, we can write $w = (w')^{-1}c$, and more generally we have $\mathcal{W}_\mathfrak{g} = \mathcal{W}_\mathfrak{h}\,\mathcal{C}$. Using this decomposition to rewrite the numerator on the right side of (15), we have

$$\begin{aligned}
\frac{\sum_{w\in\mathcal{W}_\mathfrak{g}} (-1)^w e^{iw(\boldsymbol{\lambda}-\boldsymbol{\rho}_\mathfrak{g})}}{\chi\big(\mathcal{S}^+_{L\mathfrak{h}/\mathfrak{t}}\big) - \chi\big(\mathcal{S}^-_{L\mathfrak{h}/\mathfrak{t}}\big)} &= \sum_{c\in\mathcal{C}} (-1)^c\, \frac{\sum_{w\in\mathcal{W}_\mathfrak{h}} (-1)^w e^{iwc(\boldsymbol{\lambda}-\boldsymbol{\rho}_\mathfrak{g})}}{\chi\big(\mathcal{S}^+_{L\mathfrak{h}/\mathfrak{t}}\big) - \chi\big(\mathcal{S}^-_{L\mathfrak{h}/\mathfrak{t}}\big)} \\
&= \sum_{c\in\mathcal{C}} (-1)^c\, \chi\big(\mathcal{U}_{c\,(\boldsymbol{\lambda}-\boldsymbol{\rho}_\mathfrak{g})+\boldsymbol{\rho}_\mathfrak{h}}\big),
\end{aligned}$$

where the second line follows by applying the Weyl-Kac character formula (11) for $\hat{L}\mathfrak{h}$. This proves the character form of the identity (13). ∎

The subset $\mathcal{C} \subset \mathcal{W}_\mathfrak{g}$ appearing in Theorem 4 does not depend on the weight $\boldsymbol{\lambda}$. Rather, it consists of all elements of the affine Weyl group of $\mathfrak{g}$ that map the fundamental Weyl alcove for $\mathfrak{g}$ into the fundamental Weyl alcove for $\mathfrak{h}$. Since the affine Weyl group acts simply transitively on the Weyl alcoves, it follows that the cardinality of $\mathcal{C}$ is the ratio of the volumes of the fundamental alcoves for $\mathfrak{g}$ and $\mathfrak{h}$.

Equivalently, the elements of $\mathcal{C}$ are representatives of the cosets of $\mathcal{W}_{\mathfrak{h}}$ in $\mathcal{W}_{\mathfrak{g}}$, so the cardinality of $\mathcal{C}$ is the index of $\mathcal{W}_{\mathfrak{h}}$ in $\mathcal{W}_{\mathfrak{g}}$. In particular, the sum appearing in (13) is finite if and only if $\mathfrak{h}$ is semi-simple. In such cases, $|\mathcal{C}|$ is the index of $W_{\mathfrak{h}}$ in $W_{\mathfrak{g}}$, which is also the Euler number of the corresponding homogeneous space G/H. Examples of pairs $\mathfrak{h} \subset \mathfrak{g}$ with both $\mathfrak{h}$ and $\mathfrak{g}$ semi-simple include $D_n \subset B_n$ with $|\mathcal{C}| = 2$, as well as the case $B_4 \subset F_4$ with $|\mathcal{C}| = 3$ that prompted [2]. On the other hand, for pairs $\mathfrak{h} \subset \mathfrak{g}$ corresponding to *complex* homogeneous spaces G/H, the group H must contain a U(1) component, and so (13) is an infinite sum. We note that in the physics literature (see [6, 7]), the $N = 1$ superconformal coset models on G/H possess an additional $N = 2$ symmetry precisely when $\mathcal{C}$ is infinite.

At the other extreme, if $\mathfrak{h} = \mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$, then $\boldsymbol{\rho}_{\mathfrak{h}}$ vanishes, $\mathcal{C}$ is the full affine Weyl group $\mathcal{W}_{\mathfrak{g}}$, and the homogeneous Weyl-Kac formula becomes

$$\mathcal{H}_{\boldsymbol{\lambda}} \otimes \mathcal{S}^+_{L\mathfrak{g}/L\mathfrak{t}} - \mathcal{H}_{\boldsymbol{\lambda}} \otimes \mathcal{S}^-_{L\mathfrak{g}/L\mathfrak{t}} = \sum\nolimits_{w \in \mathcal{W}_{\mathfrak{g}}} (-1)^w \, \mathcal{U}_{w(\boldsymbol{\lambda} - \boldsymbol{\rho}_{\mathfrak{g}})}. \tag{16}$$

This identity is equivalent to the Weyl-Kac character formula (11), but it is expressed slightly differently. Since $\mathfrak{t}$ is abelian, the irreducible positive energy representation $\mathcal{U}_{\boldsymbol{\mu}}$ of $\tilde{L}\mathfrak{t}$ takes the particularly simple form

$$\mathcal{U}_{\boldsymbol{\mu}} = \mathrm{Sym}^*\Big(\bigoplus\nolimits_{k>0} \mathfrak{t}_{\mathbb{C}} z^k\Big) = \bigotimes\nolimits_{k>0} \mathrm{Sym}^*\left(\mathfrak{t}_{\mathbb{C}} z^k\right),$$

where Sym^* is the symmetric algebra, and the character of this representation is

$$\chi(\mathcal{U}_{\boldsymbol{\mu}}) = e^{i\boldsymbol{\mu}} \prod\nolimits_{k>0} (1 + z^k + \cdots)^{\dim \mathfrak{t}} = e^{i\boldsymbol{\mu}} \prod\nolimits_{k>0} (1 - z^k)^{-\dim \mathfrak{t}}. \tag{17}$$

On the other hand, the signed character of the spin representation on $L\mathfrak{t}/\mathfrak{t}$ is

$$\chi\big(\mathcal{S}^+_{L\mathfrak{t}/\mathfrak{t}}\big) - \chi\big(\mathcal{S}^-_{L\mathfrak{t}/\mathfrak{t}}\big) = \prod\nolimits_{k>0} (1 - z^k)^{\dim \mathfrak{t}}, \tag{18}$$

since the product in (10) is taken over the positive roots $(k, 0, 0)$, each counted with multiplicity $\dim \mathfrak{t}$. In particular, the products in the characters (17) and (18) cancel each other, yielding the Weyl-Kac character formula for LT:

$$\chi\big(\mathcal{U}_{\boldsymbol{\mu}} \otimes \mathcal{S}^+_{L\mathfrak{t}/\mathfrak{t}}\big) - \chi\big(\mathcal{U}_{\boldsymbol{\mu}} \otimes \mathcal{S}^-_{L\mathfrak{t}/\mathfrak{t}}\big) = e^{i\boldsymbol{\mu}}.$$

So, multiplying the formula (16) by the character (18), we recover the usual form of the Weyl-Kac character formula (11) for LG.

Example. Take $\mathfrak{g} = \mathfrak{su}(2)$ and let $\mathfrak{h} = \mathfrak{u}(1)$ be the Cartan subalgebra of diagonal elements. In this particular case, we can use the homogeneous Weyl-Kac formula to explicitly compute the character of the entire spin representation $\mathcal{S}_{L\mathfrak{g}/L\mathfrak{h}}$, not just the difference of the two half-spin representations. Here we have $\boldsymbol{\rho}_{\mathfrak{g}} = (0, \rho_{\mathfrak{g}}, -c_{\mathfrak{g}}) = (0, 1, -2)$, and so the lowest weight of the spin representation is $-\boldsymbol{\rho}_{\mathfrak{g}} = (0, -1, 2)$. The half-spin representation $\mathcal{S}^+$ (resp. $\mathcal{S}^-$) is obtained by acting on a lowest weight vector by an even (resp. odd) number of Clifford multiplications by the positive generators E_+ and $E_\pm z^n$ for $n > 0$ of $L\mathfrak{g}/L\mathfrak{h}$. Since each of these generators shifts the $\mathfrak{su}(2)$ weight by ± 2, the $\mathfrak{su}(2)$ weights of all elements in $\mathcal{S}^+$ must be of the form $4n - 1$, while the weights for $\mathcal{S}^-$ are all of the form $4n + 1$. Hence the weights of $\mathcal{S}^+$ and $\mathcal{S}^-$ are distinct, and thus there is no cancellation when we take their difference.

Applying (16) for the case of the trivial representation with $\boldsymbol{\lambda} = 0$, we obtain

$$\mathcal{S}^+_{L\mathfrak{g}/L\mathfrak{h}} = \sum\nolimits_{w \in \mathcal{W}^+_{\mathfrak{g}}} \mathcal{U}_{w(0,-1,2)} = \sum\nolimits_{n \in \mathbb{Z}} \mathcal{U}_{(2n^2 - n, 4n-1, 2)},$$
$$\mathcal{S}^-_{L\mathfrak{g}/L\mathfrak{h}} = \sum\nolimits_{w \in \mathcal{W}^-_{\mathfrak{g}}} \mathcal{U}_{w(0,-1,2)} = \sum\nolimits_{n \in \mathbb{Z}} \mathcal{U}_{(2n^2 + n, 4n+1, 2)},$$

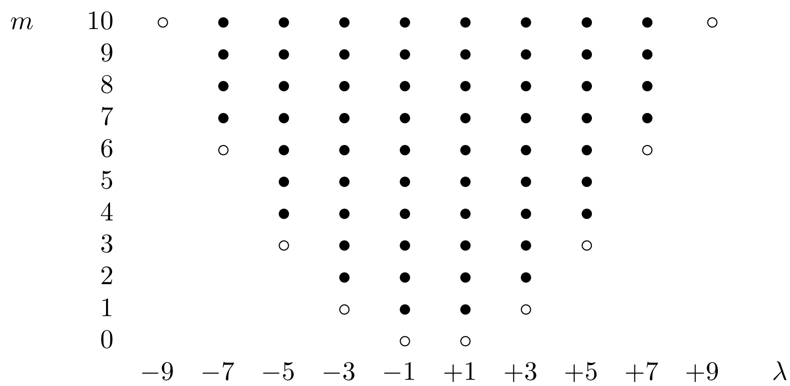

FIGURE 1. The weights of the spin representation on $L\mathrm{SU}(2)/L\mathrm{U}(1)$.

where we have explicitly written out the action of $\mathcal{W}_{\mathfrak{su}(2)} \cong \mathbb{Z}_2 \ltimes \mathbb{Z}$:

$$w_n^{\pm}(m, \lambda, h) = (m \pm \lambda n + hn^2, \pm\lambda + 2hn, h).$$

Using (17) for $\chi(\mathcal{U}_{\boldsymbol{\mu}})$, the characters of the half-spin representations are

$$\chi\big(\mathcal{S}^{+}_{L\mathfrak{g}/L\mathfrak{h}}\big)(z, w, u) = u^2 \sum\nolimits_{n\in\mathbb{Z}} w^{4n-1} z^{2n^2-n} \prod\nolimits_{k>0} (1 - z^k)^{-1},$$
$$\chi\big(\mathcal{S}^{-}_{L\mathfrak{g}/L\mathfrak{h}}\big)(z, w, u) = u^2 \sum\nolimits_{n\in\mathbb{Z}} w^{4n+1} z^{2n^2+n} \prod\nolimits_{k>0} (1 - z^k)^{-1},$$

where the powers of z, w, and u correspond to the energy, $\mathfrak{su}(2)$ weight, and level respectively. Combining these half-spin representations, the total spin representation has character

$$\chi\big(\mathcal{S}_{L\mathfrak{g}/L\mathfrak{h}}\big)(z, w, u) = u^2 \sum\nolimits_{n\in\mathbb{Z}} w^{2n-1} z^{\frac{1}{2}n(n-1)} \prod\nolimits_{k>0} (1 - z^k)^{-1}.$$

The orbit of the lowest weight $-\boldsymbol{\rho}_{\mathfrak{g}} = (0, -1, 2)$ under the affine Weyl group consists of all weights $(m, \lambda, 2)$ with λ odd and $m = \frac{1}{8}(\lambda^2 - 1)$. This equation sweeps out a parabola, and the remaining weights live inside this parabola, satisfying $m > \frac{1}{8}(\lambda^2 - 1)$. The weights of $\mathcal{S}_{L\mathfrak{g}/L\mathfrak{h}}$ are shown in Figure 1, with the orbit of $-\boldsymbol{\rho}_{\mathfrak{g}}$ drawn as open circles. The multiplicity of any such weight can be derived from (17) and is given by the number of partitions of $m - \frac{1}{8}(\lambda^2 - 1)$ into positive integers.

4. The Clifford algebra $\mathrm{Cl}(\mathfrak{g})$

Let $\mathfrak{g}$ be a finite dimensional Lie algebra with an ad-invariant inner product. Recall that the Clifford algebra $\mathrm{Cl}(\mathfrak{g})$ is generated by the elements of $\mathfrak{g}$ subject to the anti-commutator relation $\{X, Y\} = X \cdot Y + Y \cdot X = 2\langle X, Y\rangle$ for all $X, Y \in \mathfrak{g}$. There is a natural Clifford action on the exterior algebra $\Lambda^*(\mathfrak{g}^*)$, which is given on the generators $X \in \mathfrak{g}$ by $c(X) = \iota_X + \varepsilon_{X^*}$, where ι_X is interior contraction by $X \in \mathfrak{g}$ and ε_{X^*} is exterior multiplication by the dual element $X^* \in \mathfrak{g}^*$ satisfying $X^*(Y) = \langle X, Y\rangle$. Using the distinguished element 1 of the exterior algebra, the map $x \mapsto c(x)1$ gives an isomorphism $\mathrm{Cl}(\mathfrak{g}) \to \Lambda^*(\mathfrak{g}^*)$ of left $\mathrm{Cl}(\mathfrak{g})$-modules, called the Chevalley identification. We may therefore view the Clifford algebra as the exterior algebra $\Lambda^*(\mathfrak{g}^*)$ with the alternative multiplication

$$X^* \cdot \eta = X^* \wedge \eta + \iota_X \eta \tag{19}$$

for $X \in \mathfrak{g}$ and $\eta \in \Lambda^*(\mathfrak{g}^*)$.

Consider the graded Lie superalgebra $\hat{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathbb{R}_1$, where the subscript denotes the integer grading. The exterior algebra $\Lambda^*(\mathfrak{g}^*)$ is a representation of this Lie superalgebra $\hat{\mathfrak{g}}$, with $\mathfrak{g}_{-1}$ acting by interior contraction, $\mathfrak{g}_0$ acting by the coadjoint action, and the generator $d \in \mathbb{R}_1$ acting as the exterior derivative. On the generators $\xi \in \mathfrak{g}^*$, these operators are given by

$$\begin{aligned} \iota_X \xi &= \xi(X) & \iota_X &: \Lambda^k(\mathfrak{g}^*) \to \Lambda^{k-1}(\mathfrak{g}^*) \\ (\operatorname{ad}^*_X \xi)(Y) &= -\xi(\operatorname{ad}_X Y) & \operatorname{ad}^*_X &: \Lambda^k(\mathfrak{g}^*) \to \Lambda^{k+0}(\mathfrak{g}^*) \\ (d\xi)(X,Y) &= -\tfrac{1}{2}\,\xi([X,Y]) & d &: \Lambda^k(\mathfrak{g}^*) \to \Lambda^{k+1}(\mathfrak{g}^*) \end{aligned}$$

for $X, Y \in \mathfrak{g}$. These operators then extend as super-derivations to the full exterior algebra, and they satisfy the identities $[\operatorname{ad}^*_X, \iota_Y] = \iota_{[X,Y]}$ and $\{d, \iota_X\} = \operatorname{ad}^*_X$. If we perturb this action by taking $d' = d - \iota_{\Omega^*}$, where Ω is a closed $\mathfrak{g}$-invariant form of odd degree, then the commutation relations on $\hat{\mathfrak{g}}$ are unchanged.

Theorem 5. *Using the Chevalley identification, the action of $\hat{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathbb{R}_1$ on $\Lambda^*(\mathfrak{g}^*)$ can be expressed in terms of the adjoint action of the Clifford algebra as*

$$\operatorname{ad} X^* = 2\,\iota_X \qquad X^* \in \Lambda^1(\mathfrak{g}^*) \tag{20}$$

$$\operatorname{ad} dX^* = 2\operatorname{ad}^*_X \qquad dX^* \in \Lambda^2(\mathfrak{g}^*) \tag{21}$$

$$\operatorname{ad}\Omega = 2\,d - 2\,\iota_{\Omega^*} \qquad \Omega \in \Lambda^3(\mathfrak{g}^*) \tag{22}$$

where Ω is the fundamental 3-form given by $\Omega(X,Y,Z) = -\frac{1}{6}\langle X, [Y,Z]\rangle$.

Proof. First, we show that the operators ι_X and ad^*_X are super-derivations with respect to the Clifford multiplication (19). For $X, Y \in \mathfrak{g}$ and $\eta \in \Lambda^*(\mathfrak{g}^*)$, we have

$$\begin{aligned} \iota_X(Y^* \cdot \eta) &= \iota_X(Y^* \wedge \eta) + \iota_X \iota_Y \eta \\ &= (\iota_X Y^*) \wedge \eta - Y^* \wedge \iota_X \eta - \iota_Y \iota_X \eta \\ &= (\iota_X Y^*) \cdot \eta - Y^* \cdot \iota_X \eta, \\ \operatorname{ad}^*_X(Y^* \cdot \eta) &= \operatorname{ad}^*_X(Y^* \wedge \eta) + \operatorname{ad}^*_X \iota_Y\, \eta \\ &= (\operatorname{ad}^*_X Y^*) \wedge \eta + Y^* \wedge \operatorname{ad}^*_X \eta + \iota_Y \operatorname{ad}^*_X \eta + \iota_{[X,Y]}\eta \\ &= (\operatorname{ad}^*_X Y^*) \cdot \eta + Y^* \cdot \operatorname{ad}^*_X \eta. \end{aligned}$$

Now, to prove the identities (20) and (21), we need only verify them for the generators $\mathfrak{g}^* = \Lambda^1(\mathfrak{g}^*)$, but it follows from the definition of the Clifford algebra that

$$\{X^*, Y^*\} = 2\,\langle X, Y\rangle = 2\,\iota_X Y^*,$$

and by applying (20) and the identity $\{d, \iota_X\} = \operatorname{ad}^*_X$, we obtain

$$[dX^*, Y^*] = -2\,\iota_Y dX^* = -2\operatorname{ad}^*_Y X^* = 2\operatorname{ad}^*_X Y^*.$$

To prove (22), we first verify that it holds when acting on a generator $X^* \in \mathfrak{g}^*$:

$$\{\Omega, X^*\}(Y,Z) = (2\,\iota_X\,\Omega)(Y,Z) = -\langle X, [Y,Z]\rangle = (2\,dX^*)(Y,Z).$$

Finally we show that $d' = d - \iota_{\Omega^*}$ is a super-derivation for Clifford multiplication:

$$\begin{aligned} d'(X^* \cdot \eta) &= d(X^* \wedge \eta) - \iota_{\Omega^*}(X^* \wedge \eta) + d\,\iota_X\,\eta - \iota_{\Omega^*}\,\iota_X\,\eta \\ &= (dX^*) \wedge \eta - X^* \wedge d\eta - \iota_{(dX^*)^*}\,\eta + X^* \wedge \iota_{\Omega^*}\eta \\ &\qquad - \iota_X d\eta + \operatorname{ad}^*_X \eta + \iota_X \iota_{\Omega^*}\,\eta \\ &= (d'X^*) \cdot \eta - X^* \cdot d'\eta, \end{aligned}$$

where we use the expansion $(d'X^*)\cdot\eta = (dX^*)\cdot\eta = (dX^*)\wedge\eta + \mathrm{ad}^*_X\,\eta - \iota_{(dX^*)^*}\,\eta$. $\square$

Although the Clifford algebra $\mathrm{Cl}(\mathfrak{g})$ does not admit an integer grading, it does have the distinguished subspaces $\mathfrak{g}$ and $\mathfrak{spin}(\mathfrak{g})$, which correspond via the Chevalley identification to the first two degrees of the exterior algebra:

$$\Lambda^1(\mathfrak{g}^*) \longleftrightarrow \mathfrak{g} \subset \mathrm{Cl}(\mathfrak{g}), \qquad \Lambda^2(\mathfrak{g}^*) \longleftrightarrow \mathfrak{spin}(\mathfrak{g}) \subset \mathrm{Cl}(\mathfrak{g}).$$

Since $\mathrm{Spin}(\mathfrak{g})$ is the double cover of $\mathrm{SO}(\mathfrak{g})$, there is a Lie algebra isomorphism $\mathfrak{spin}(\mathfrak{g}) \cong \mathfrak{so}(\mathfrak{g})$, and given any element $a \in \mathfrak{so}(\mathfrak{g})$, the corresponding element of $\tilde{a} \in \mathfrak{spin}(\mathfrak{g})$ is uniquely determined by the identity $[\tilde{a}, X^*] = (aX)^*$ for all $X \in \mathfrak{g}$. In particular, the adjoint action $\mathrm{ad} : \mathfrak{g} \to \mathfrak{so}(\mathfrak{g})$ lifts to a Lie algebra homomorphism $\widetilde{\mathrm{ad}} : \mathfrak{g} \to \mathfrak{spin}(\mathfrak{g})$ satisfying

$$[\widetilde{\mathrm{ad}}\,X, Y^*] = (\mathrm{ad}_X\,Y)^* = \mathrm{ad}^*_X\,Y^*.$$

However, from the identity (21), we see that the spin lift of the adjoint action must be $\widetilde{\mathrm{ad}}\,X = \frac{1}{2}\,dX^*$.

Let $\{X_i\}$ be a basis of $\mathfrak{g}$, and let $\{X_i^*\}$ denote the corresponding dual basis of $\mathfrak{g}$ satisfying $\langle X_i^*, X_j\rangle = \delta_{ij}$. In terms of this basis, the map $\widetilde{\mathrm{ad}} : X \mapsto \frac{1}{2}\,dX^*$ is

$$\widetilde{\mathrm{ad}}\,X = -\frac{1}{4}\sum\nolimits_i X_i^* \cdot [X, X_i], \tag{23}$$

while the element $\gamma = \frac{1}{4}\,\Omega$ corresponding to the fundamental 3-form is given by

$$\gamma = -\frac{1}{24}\sum\nolimits_{i,j} X_i^* \cdot X_j^* \cdot [X_i, X_j] = \frac{1}{6}\sum\nolimits_i X_i^* \cdot \widetilde{\mathrm{ad}}\,X_i. \tag{24}$$

Rewriting Theorem 5 in terms of this new notation, we obtain the following:

Corollary 6. *The elements* 1, X, $\widetilde{\mathrm{ad}}\,X$, γ *for* $X \in \mathfrak{g}$ *span a Lie superalgebra* $\mathbb{R}_+ \oplus \mathfrak{g}_- \oplus \mathfrak{g}_+ \oplus \mathbb{R}_-$ *in the Clifford algebra* $\mathrm{Cl}(\mathfrak{g})$ *with the commutation relations*

$$\begin{aligned} &[\widetilde{\mathrm{ad}}\,X, Y] = [X, Y], && [\widetilde{\mathrm{ad}}\,X, \widetilde{\mathrm{ad}}\,Y] = \widetilde{\mathrm{ad}}[X, Y], && [\widetilde{\mathrm{ad}}\,X, \gamma] = 0,\\ &\{X, Y\} = 2\,\langle X, Y\rangle, && \{\gamma, X\} = \widetilde{\mathrm{ad}}\,X, && \{\gamma, \gamma\} = -\tfrac{1}{24}\,\mathrm{tr}_{\mathfrak{g}}\,\Delta^{\mathfrak{g}}_{\mathrm{ad}}, \end{aligned}$$

where $\Delta^{\mathfrak{g}}_{\mathrm{ad}} = -\frac{1}{2}\sum_i \mathrm{ad}_{X_i^*}\,\mathrm{ad}_{X_i}$ *is the quadratic Casimir operator.*

Proof. All of the commutation relations follow immediately from Theorem 5 and the above discussion with the exception of that for $\{\gamma, \gamma\}$. For example, we derive

$$\begin{aligned} [\widetilde{\mathrm{ad}}\,X, \widetilde{\mathrm{ad}}\,Y] &= \tfrac{1}{2}\,\mathrm{ad}^*_X\,dY^* = \tfrac{1}{2}\,d\,\mathrm{ad}^*_X\,Y^* = \widetilde{\mathrm{ad}}[X, Y],\\ [\widetilde{\mathrm{ad}}\,X, \gamma] &= -\big[\tfrac{1}{4}\Omega, \tfrac{1}{2}dX^*\big] = -\tfrac{1}{4}ddX^* = 0. \end{aligned}$$

To compute $\{\gamma, \gamma\}$, we note that the fundamental 3-form is closed, so we have

$$\{\gamma, \gamma\} = \big\{\tfrac{1}{4}\,\Omega, \tfrac{1}{4}\,\Omega\big\} = \tfrac{1}{8}\,d\Omega - \tfrac{1}{8}\,\iota_{\Omega^*}\,\Omega = -\tfrac{1}{8}\,\langle\Omega, \Omega\rangle.$$

Written in terms of an orthonormal basis $\{X_i\}$ for $\mathfrak{g}$, the fundamental 3-form is

$$\Omega = -\sum\nolimits_{i<j<k}\big\langle X_i, [X_j, X_k]\big\rangle\, X_i^* \wedge X_j^* \wedge X_k^*,$$

and so its norm is given by

$$\begin{aligned} \langle\Omega, \Omega\rangle &= \frac{1}{6}\sum\nolimits_{i,j,k}\big\langle X_i, [X_j, X_k]\big\rangle^2 = \frac{1}{6}\sum\nolimits_{j,k}\big\langle [X_j, X_k], [X_j, X_k]\big\rangle\\ &= -\frac{1}{6}\sum\nolimits_{j,k}\big\langle X_k, [X_j, [X_j, X_k]]\big\rangle = \frac{1}{3}\,\mathrm{tr}_{\mathfrak{g}}\,\Delta^{\mathfrak{g}}_{\mathrm{ad}}, \end{aligned}$$

which yields the desired anti-commutator $\{\gamma, \gamma\} = -\frac{1}{24}\,\mathrm{tr}_{\mathfrak{g}}\,\Delta^{\mathfrak{g}}_{\mathrm{ad}}$. $\square$

Note that the map $\widetilde{\mathrm{ad}} : \mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ is not necessarily injective; rather its kernel is the center of $\mathfrak{g}$. So, Corollary 6 actually gives us an inclusion of the superalgebra

$$\tilde{\mathfrak{g}} := \mathbb{R} \oplus \mathfrak{g} \oplus [\mathfrak{g},\mathfrak{g}] \oplus \langle \mathfrak{g}, [\mathfrak{g},\mathfrak{g}]\rangle \subset \Lambda^*(\mathfrak{g}^*) \cong \mathrm{Cl}(\mathfrak{g}) \tag{25}$$

into the Clifford algebra of $\mathfrak{g}$. Also note that this Lie superalgebra $\tilde{\mathfrak{g}}$, with the commutation relations given by Corollary 6, is the quantized form of the graded Lie superalgebra $\hat{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathbb{R}_1$ discussed above.

5. The Dirac operator on $\mathfrak{g}$

Let $\mathfrak{g}$ be a Lie algebra with an ad-invariant inner product, and let $\mathbb{S}_\mathfrak{g}$ be the complex spin representation of the Clifford algebra $\mathrm{Cl}(\mathfrak{g})$. If $\mathfrak{g}$ is even dimensional then we have $\mathrm{Cl}(\mathfrak{g}) \otimes \mathbb{C} \cong \mathrm{End}(\mathbb{S}_\mathfrak{g})$, and in general the spin representation $\mathrm{Cl}(\mathfrak{g}) \hookrightarrow \mathrm{End}(\mathbb{S}_\mathfrak{g})$ is faithful. To simplify our notation, in the following we implicitly identify $\mathrm{Cl}(\mathfrak{g})$ with its image in $\mathrm{End}(\mathbb{S}_\mathfrak{g})$ under the spin representation. We recall from the previous section that the adjoint action ad of $\mathfrak{g}$ on itself lifts to the representation $\widetilde{\mathrm{ad}} : \mathfrak{g} \to \mathrm{Cl}(\mathfrak{g})$ given by (23).

Let V be an arbitrary $\mathfrak{g}$-module where the $\mathfrak{g}$-action is the map $r : \mathfrak{g} \to \mathrm{End}(V)$. Alternatively, this representation r may be viewed as the $\mathrm{End}(V)$-valued 1-form $\hat{r} \in \mathrm{End}(V) \otimes \Lambda^*(\mathfrak{g}^*)$ given tautologically by $\hat{r}(X) = r(X)$ for all $X \in \mathfrak{g}$. Identifying $\Lambda^*(\mathfrak{g}^*)$ with $\mathrm{Cl}(\mathfrak{g})$ via the Chevalley map, the element $\hat{r} \in \mathrm{End}(V) \otimes \mathrm{Cl}(\mathfrak{g})$ becomes an operator on the tensor product $V \otimes \mathbb{S}_\mathfrak{g}$. Perturbing this operator slightly we define the Dirac operator $\partial\!\!\!/_r$ on $V \otimes \mathbb{S}_\mathfrak{g}$ to be the element

$$\partial\!\!\!/_r := \hat{r} + 1 \otimes \tfrac{1}{2}\,\Omega \,\in\, \mathrm{End}(V) \otimes \mathrm{Cl}(\mathfrak{g}),$$

where $\Omega \in \mathrm{Cl}(\mathfrak{g})$ is the cubic term given by (22). Written in terms of a basis $\{X_i\}$ for $\mathfrak{g}$ and the dual basis $\{X_i^*\}$ satisfying $\langle X_i^*, X_j\rangle = \delta_{i,j}$, this Dirac operator is

$$\begin{aligned} \partial\!\!\!/_r &= \sum_i X_i^*\, r(X_i) - \frac{1}{12} \sum_{i,j} X_i^* \cdot X_j^* \cdot [X_i, X_j] \\ &= \sum_i X_i^* \Big(r(X_i) + \frac{1}{3}\, \widetilde{\mathrm{ad}}\, X_i \Big). \end{aligned}$$

Note that the second form of this operator resembles a geometric Dirac operator for the connection $\nabla_X = r(X) + \frac{1}{3}\,\widetilde{\mathrm{ad}}\, X$. Indeed, if r is the right action of $\mathfrak{g}$ on functions, then this is the *reductive connection* on the spin bundle over G (see [12]).

Rather than choosing a particular representation V, we can instead take r to be the canonical inclusion $r : \mathfrak{g} \hookrightarrow U(\mathfrak{g})$ of $\mathfrak{g}$ into its universal enveloping algebra $U(\mathfrak{g})$. This gives us a universal Dirac operator $\partial\!\!\!/$, which is an element of the non-abelian Weil algebra $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})$, introduced by Alekseev and Meinrenken in [1]. Again identifying $\mathrm{Cl}(\mathfrak{g})$ with $\Lambda^*(\mathfrak{g}^*)$, the element $\partial\!\!\!/$ is characterized by the identity

$$\iota_X\, \partial\!\!\!/ = \varrho(X) := r(X) \otimes 1 + 1 \otimes \widetilde{\mathrm{ad}}\, X \tag{26}$$

for all $X \in \mathfrak{g}$, where $\varrho = r \otimes 1 + 1 \otimes \widetilde{\mathrm{ad}}$ is the diagonal action of $\mathfrak{g}$ on $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})$. Squaring the Dirac operator, we obtain the Weitzenböck formula

$$\begin{aligned} \partial\!\!\!/^2 &= \hat{r} \cdot \hat{r} + \big\{ \hat{r}, \tfrac{1}{2}\,\Omega \big\} + \tfrac{1}{2}\,\Omega \cdot \tfrac{1}{2}\,\Omega \\ &= \langle \hat{r}, \hat{r} \rangle + \hat{r} \wedge \hat{r} + d\hat{r} + \tfrac{1}{8}\,\{\Omega, \Omega\} = -2\,\Delta_r^\mathfrak{g} - \tfrac{1}{12}\,\mathrm{tr}_\mathfrak{g}\,\Delta_{\mathrm{ad}}^\mathfrak{g}, \end{aligned} \tag{27}$$

where the "curvature" term $d\hat{r} + \hat{r} \wedge \hat{r}$ vanishes since

$$(d\hat{r} + \hat{r} \wedge \hat{r})(X, Y) = \tfrac{1}{2}\big(-r([X,Y]) + [r(X), r(Y)]\,\big) = 0.$$

Note that the square of the Dirac operator has no Clifford algebra component and is thus an element $\partial\!\!\!/^2 \in U(\mathfrak{g})$ of the universal enveloping algebra. In fact, since the Casimir operator commutes with the $\mathfrak{g}$-action, the element $\partial\!\!\!/^2$ lies in the center of $U(\mathfrak{g})$. Considering the Dirac operator itself, given any $X \in \mathfrak{g}$ we have

$$[\varrho(X), \partial\!\!\!/] = [\iota_X \partial\!\!\!/, \partial\!\!\!/] = \iota_X(\partial\!\!\!/ \cdot \partial\!\!\!/) = 0, \tag{28}$$

and thus $\partial\!\!\!/$ is invariant under the diagonal action ϱ of $\mathfrak{g}$ on $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})$. We can summarize the above results by stating that the Lie superalgebra

$$\mathbb{R} \,\oplus\, (1 \otimes \mathfrak{g}) \,\oplus\, \varrho(\mathfrak{g}) \,\oplus\, \mathbb{R}\partial\!\!\!/ \,\oplus\, \mathbb{R}\Delta^{\mathfrak{g}} \subset U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})$$

is a central extension of the Lie superalgebra $\tilde{\mathfrak{g}}$ from Corollary 6 by the span of the quadratic Casimir operator $\Delta^{\mathfrak{g}}$. The commutation relations in this extension are the same as in Corollary 6, with the exception of the square of the Dirac operator which is given by (27). To obtain the corresponding "classical" algebra, we let this superalgebra act on the non-abelian Weil algebra via the adjoint action. Since the elements 1 and $\Delta^{\mathfrak{g}}$ lie in the center of the universal enveloping algebra, we are left with the graded Lie superalgebra $\hat{\mathfrak{g}}$, as Alekseev and Meinrenken show in [1].

Theorem 7. *The non-abelian Weil algebra $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})$ is a representation of the graded Lie superalgebra $\hat{\mathfrak{g}} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathbb{R}_1$ spanned by the operators ι_X, $\mathcal{L}_X$, d for $X \in \mathfrak{g}$ given by*

$$\iota_X = \mathrm{ad}\big(\tfrac{1}{2}X\big), \qquad \mathcal{L}_X = \mathrm{ad}\big(\varrho(X)\big), \qquad d = \mathrm{ad}\big(\partial\!\!\!/\big).$$

Now, suppose that $\mathfrak{g}$ is reductive. If V_λ is the irreducible representation of $\mathfrak{g}$ with highest weight λ, then the value of the quadratic Casimir operator $\Delta^{\mathfrak{g}}_\lambda$ on V_λ is

$$\Delta^{\mathfrak{g}}_\lambda = \tfrac{1}{2}\big(\|\lambda + \rho_{\mathfrak{g}}\|^2 - \|\rho_{\mathfrak{g}}\|^2\big).$$

In addition, for reductive Lie algebras we have the identity $\frac{1}{12}\,\mathrm{tr}_{\mathfrak{g}}\, \Delta^{\mathfrak{g}}_{\mathrm{ad}} = \|\rho_{\mathfrak{g}}\|^2$, and it follows that the square of the Dirac operator $\partial\!\!\!/_\lambda$ acting on $V_\lambda \otimes \mathbb{S}_{\mathfrak{g}}$ is simply the constant $\partial\!\!\!/^2_\lambda = -\|\lambda + \rho_{\mathfrak{g}}\|^2$.

6. The Dirac operator on $\mathfrak{g}/\mathfrak{h}$

Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$, and let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{h}$ with respect to the ad-invariant inner product on $\mathfrak{g}$. The adjoint action of $\mathfrak{h}$ on $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ respects this decomposition, so we obtain $\mathfrak{h}$-representations $\mathrm{ad}_{\mathfrak{h}}$ and $\mathrm{ad}_{\mathfrak{p}}$ on $\mathfrak{h}$ and $\mathfrak{p}$ respectively. The Clifford algebra also decomposes into the product $\mathrm{Cl}(\mathfrak{g}) \cong \mathrm{Cl}(\mathfrak{h}) \otimes \mathrm{Cl}(\mathfrak{p})$ of two Clifford algebras, and with it the spin lift of the adjoint action becomes the sum $\widetilde{\mathrm{ad}}_{\mathfrak{g}} = \widetilde{\mathrm{ad}}_{\mathfrak{h}} \otimes 1 + 1 \otimes \widetilde{\mathrm{ad}}_{\mathfrak{p}}$ of separate spin actions $\widetilde{\mathrm{ad}}_{\mathfrak{h}} : \mathfrak{h} \to \mathrm{Cl}(\mathfrak{h})$ and $\widetilde{\mathrm{ad}}_{\mathfrak{p}} : \mathfrak{h} \to \mathrm{Cl}(\mathfrak{p})$. The spin representations $\mathbb{S}_{\mathfrak{h}}$ and $\mathbb{S}_{\mathfrak{p}}$ of $\mathrm{Cl}(\mathfrak{h})$ and $\mathrm{Cl}(\mathfrak{p})$ are therefore representations of $\mathfrak{h}$, and if one or both of $\mathfrak{h}$ or $\mathfrak{p}$ is even dimensional, then we have $\mathbb{S}_{\mathfrak{g}} \cong \mathbb{S}_{\mathfrak{h}} \otimes \mathbb{S}_{\mathfrak{p}}$.

Let $\partial\!\!\!/_{\mathfrak{g}} \in U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})$ denote the universal Dirac operator on $\mathfrak{g}$ discussed in the previous section. Now consider the twisted Dirac operator $\partial\!\!\!/'_{\mathfrak{h}}$ on $\mathfrak{h}$ given by

$$\partial\!\!\!/'_{\mathfrak{h}} := \hat{r}'_{\mathfrak{h}} + \tfrac{1}{2}\,\Omega_{\mathfrak{h}} \in \big(U(\mathfrak{h}) \otimes \mathrm{Cl}(\mathfrak{p})\big) \otimes \mathrm{Cl}(\mathfrak{h}) \cong U(\mathfrak{h}) \otimes \mathrm{Cl}(\mathfrak{g})$$

where $r' = r \otimes 1 + 1 \otimes \widetilde{\mathrm{ad}}_{\mathfrak{p}}$ is the diagonal action of $\mathfrak{h}$ on $U(\mathfrak{h}) \otimes \mathrm{Cl}(\mathfrak{p})$. In other words, given any representation U of $\mathfrak{h}$, this twisted Dirac operator $\partial\!\!\!/'_{\mathfrak{h}}$ is the usual

Dirac operator $\partial\!\!\!/_{\mathfrak{h}}$ acting on the twisted space $(U \otimes \mathbb{S}_{\mathfrak{p}}) \otimes \mathbb{S}_{\mathfrak{h}} \cong U \otimes \mathbb{S}_{\mathfrak{g}}$. As we saw in (26), this Dirac operator $\partial\!\!\!/'_{\mathfrak{h}}$ is characterized by the identity

$$\iota_Z \partial\!\!\!/'_{\mathfrak{h}} = \varrho'_{\mathfrak{h}}(Z) = \varrho_{\mathfrak{g}}(Z) = \iota_Z \partial\!\!\!/_{\mathfrak{g}}$$

for all $Z \in \mathfrak{h}$, where $\varrho'_{\mathfrak{h}}$ is the diagonal action of $\mathfrak{h}$ on $\big(U(\mathfrak{h}) \otimes \mathrm{Cl}(\mathfrak{p})\big) \otimes \mathrm{Cl}(\mathfrak{h})$. Note that $\varrho'_{\mathfrak{h}}$ is just the restriction to $\mathfrak{h}$ of the diagonal action $\varrho_{\mathfrak{g}}$ of $\mathfrak{g}$ on $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})$. It then follows from (28) that the element $\partial\!\!\!/'_{\mathfrak{h}}$ commutes with the diagonal action $\varrho'_{\mathfrak{h}}$.

Define $\partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}} = \partial\!\!\!/_{\mathfrak{g}} - \partial\!\!\!/'_{\mathfrak{h}}$ to be the difference of these two operators. This element $\partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}}$ is then *basic* with respect to $\hat{\mathfrak{h}}$, or in other words it satisfies the identities

$$\iota_Z\, \partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}} = 0, \qquad \mathcal{L}_Z\, \partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}} = \big[\varrho'_h(Z), \partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}}\big] = 0,$$

for all $Z \in \mathfrak{h}$. This Dirac operator can also be written as the element

$$\partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}} = \hat{r}_{\mathfrak{p}} + 1 \otimes \tfrac{1}{2}\,\Omega_{\mathfrak{p}} \in \big(U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{p})\big)^{\mathfrak{h}},$$

where $\hat{r}_{\mathfrak{p}} \in U(\mathfrak{g}) \otimes \Lambda^1(\mathfrak{p}^*)$ corresponds to the map $r : \mathfrak{p} \hookrightarrow U(\mathfrak{g})$, and $\Omega_{\mathfrak{p}} \in \Lambda^3(\mathfrak{p}^*)$ is the fundamental 3-form given by $\Omega_{\mathfrak{p}}(X,Y,Z) = -\frac{1}{6}\langle X, [Y,Z]\rangle$ for all $X, Y, Z \in \mathfrak{p}$. To keep track of the cubic terms, note that the fundamental 3-form decomposes as

$$\Omega_{\mathfrak{g}} = \Omega_{\mathfrak{h}} + \Omega_{\mathfrak{p}} + 3\,\Omega_{\mathfrak{h}\mathfrak{p}\mathfrak{p}},$$

into its projections onto $\Lambda^3(\mathfrak{h}^*)$, $\Lambda^3(\mathfrak{p}^*)$, and $\mathfrak{h}^* \wedge \mathfrak{p}^* \wedge \mathfrak{p}^*$ respectively. This extra contribution corresponds to the twisted term $\hat{r}'_{\mathfrak{h}} - \hat{r}_{\mathfrak{h}} = \frac{3}{2}\,\Omega_{\mathfrak{h}\mathfrak{p}\mathfrak{p}}$ appearing in $\partial\!\!\!/'_{\mathfrak{h}}$.

If $\{X_i\}$ and $\{X_i^*\}$ are dual bases for $\mathfrak{p}$, then the Dirac operator $\partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}}$ is

$$\begin{aligned}\partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}} &= \sum_i X_i^*\, r(X_i) - \frac{1}{12}\sum_{i,j} X_i^* \cdot X_j^* \cdot [X_i, X_j]_{\mathfrak{p}} \\ &= \sum_i X_i^* \Big(r(X_i) + \frac{1}{3}\,\widetilde{\mathrm{ad}}_{\mathfrak{p}}\, X_i\Big),\end{aligned}$$

where $[X,Y]_{\mathfrak{p}}$ for $X, Y \in \mathfrak{p}$ denotes the projection of $[X,Y]$ onto $\mathfrak{p}$, and

$$\widetilde{\mathrm{ad}}_{\mathfrak{p}}\, X = -\frac{1}{4}\sum\nolimits_i X_i^* \cdot [X, X_i]_{\mathfrak{p}}$$

for $X \in \mathfrak{p}$. (Note that all of the sums here are taken over a basis of $\mathfrak{p}$, not of $\mathfrak{g}$.) The geometric version of this Dirac operator $\partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}}$, viewed as an operator on twisted spinors on the homogeneous space G/H, is discussed in [13, 14] and [9].

To compute the square of $\partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}}$, we first show that $\partial\!\!\!/'_{\mathfrak{h}}$ and $\partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}}$ decouple,

$$\big\{\partial\!\!\!/'_{\mathfrak{h}}, \partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}}\big\} = \big\{\hat{r}'_{\mathfrak{h}}, \partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}}\big\} + \big\{\tfrac{1}{2}\,\Omega_{\mathfrak{h}}, \partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}}\big\} = [r'(\cdot), \partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}}]^{\tilde{}} + d_{\mathfrak{h}}\, \partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}} = 0.$$

We therefore have

$$\partial\!\!\!/^2_{\mathfrak{g}/\mathfrak{h}} = (\partial\!\!\!/_{\mathfrak{g}})^2 - (\partial\!\!\!/'_{\mathfrak{h}})^2 = -2\big(\Delta^{\mathfrak{g}}_r - \Delta^{\mathfrak{h}}_{r'}\big) - \tfrac{1}{12}\big(\mathrm{tr}_{\mathfrak{g}}\, \Delta^{\mathfrak{g}}_{\mathrm{ad}} - \mathrm{tr}_{\mathfrak{h}}\, \Delta^{\mathfrak{h}}_{\mathrm{ad}}\big).$$

Suppose that both $\mathfrak{g}$ and $\mathfrak{h}$ are reductive. If $r : \mathfrak{g} \to \mathrm{End}(V_\lambda)$ is the irreducible representation of $\mathfrak{g}$ with highest weight λ, then $\partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}}$ is an $\mathfrak{h}$-invariant operator on $V_\lambda \otimes \mathbb{S}_{\mathfrak{p}}$. Its square then takes the value

$$\partial\!\!\!/^2_{\mathfrak{g}/\mathfrak{h}}|_\mu = -\|\lambda + \rho_{\mathfrak{g}}\|^2 + \|\mu + \rho_{\mathfrak{h}}\|^2 \tag{29}$$

on the $\mathfrak{h}$-invariant subspace of $V_\lambda \otimes \mathbb{S}_{\mathfrak{p}}$ transforming like the irreducible representation U_μ of $\mathfrak{h}$ with highest weight μ. It follows that the kernel of $\partial\!\!\!/^2_{\mathfrak{g}/\mathfrak{h}}$, which is in turn the kernel of the Dirac operator $\partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}}$ itself, consists of all $\mathfrak{h}$-invariant subspaces of $V_\lambda \otimes \mathbb{S}_{\mathfrak{p}}$ transforming like U_μ, where μ is a dominant weight of $\mathfrak{h}$ satisfying

$\|\mu+\rho_{\mathfrak{h}}\|^2 = \|\lambda+\rho_{\mathfrak{g}}\|^2$. As we show in the following theorem, these subspaces are precisely the multiplet of $\mathfrak{h}$-representations corresponding to the $\mathfrak{g}$-representation V_λ, which we discussed in §3.

Theorem 8. *Let $\mathfrak{g}$ be a semi-simple Lie algebra with a maximal rank reductive Lie subalgebra $\mathfrak{h}$, and let V_λ and U_μ denote the irreducible representations of $\mathfrak{g}$ and $\mathfrak{h}$ with highest weights λ and μ. The kernel of the Dirac operator $\partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}}$ on $V_\lambda \otimes \mathbb{S}_{\mathfrak{p}}$ is*

$$\operatorname{Ker} \partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}} = \bigoplus\nolimits_{c\in C} U_{c\bullet\lambda},$$

where $c\bullet\lambda = c(\lambda+\rho_{\mathfrak{g}}) - \rho_{\mathfrak{h}}$, and $C \subset W_{\mathfrak{g}}$ is the subset of Weyl elements which map the positive Weyl chamber for $\mathfrak{g}$ into the positive Weyl chamber for $\mathfrak{h}$.

Proof. Since the Weyl group acts by isometries, the weights $c\bullet\lambda$ satisfy the identity

$$\|(c\bullet\lambda)+\rho_{\mathfrak{h}}\|^2 = \|c(\lambda+\rho_{\mathfrak{g}})\|^2 = \|\lambda+\rho_{\mathfrak{g}}\|^2,$$

and it follows from (29) that the Dirac operator $\partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}}$ vanishes on any $\mathfrak{h}$-invariant subspace of $V_\lambda\otimes\mathbb{S}_{\mathfrak{p}}$ transforming like $U_{c\bullet\lambda}$. To complete the proof, it remains to show that each of these representations occurs exactly once in the domain of the Dirac operator and that no other $\mathfrak{h}$-representations appear in its kernel. We establish these facts in the following two lemmas (see also [3]). □

Lemma 9. *For each $c\in C$, the irreducible representation $U_{c\bullet\lambda}$ of $\mathfrak{h}$ with highest weight $c\bullet\lambda = c(\lambda+\rho_{\mathfrak{g}})-\rho_{\mathfrak{h}}$ occurs exactly once in the decomposition of $V_\lambda\otimes\mathbb{S}_{\mathfrak{p}}$.*

Proof. The highest weight space of an irreducible representation of $\mathfrak{g}$ is always one dimensional, so the weight λ appears with multiplicity 1 in V_λ. Now consider the complex spin representation $\mathbb{S}_{\mathfrak{g}/\mathfrak{t}}$ associated to the orthogonal complement of a Cartan subalgebra $\mathfrak{t}$ in $\mathfrak{g}$. Given a positive root system for $\mathfrak{g}$, the character of this spin representation is $\chi(\mathbb{S}_{\mathfrak{g}/\mathfrak{t}}) = \prod_{\alpha>0}(e^{i\alpha/2}+e^{-i\alpha/2})$, and so the highest weight $\rho_{\mathfrak{g}} = \frac{1}{2}\sum_{\alpha>0}\alpha$ of $\mathbb{S}_{\mathfrak{g}/\mathfrak{t}}$ appears with multiplicity 1. The highest weight of the tensor product $V_\lambda\otimes\mathbb{S}_{\mathfrak{g}/\mathfrak{t}}$ is then $\lambda+\rho_{\mathfrak{g}}$, appearing with multiplicity 1, and likewise the weights $w(\lambda+\rho_{\mathfrak{g}})$ for $w\in W_{\mathfrak{g}}$ all have multiplicity 1 in $V_\lambda\otimes\mathbb{S}_{\mathfrak{g}/\mathfrak{t}}$.

Choosing a common Cartan subalgebra $\mathfrak{t}\subset\mathfrak{h}\subset\mathfrak{g}$, the spin representation factors as $\mathbb{S}_{\mathfrak{g}/\mathfrak{t}} \cong \mathbb{S}_{\mathfrak{p}}\otimes\mathbb{S}_{\mathfrak{h}/\mathfrak{t}}$. As we noted above, the weight $\rho_{\mathfrak{h}}$ appears with multiplicity 1 in the second factor $\mathbb{S}_{\mathfrak{h}/\mathfrak{t}}$. It follows that the weights $w\bullet\lambda$ for $w\in W_{\mathfrak{g}}$ can appear at most once in the tensor product $V_\lambda\otimes\mathbb{S}_{\mathfrak{p}}$, as each such weight contributes one weight of the form $(w\bullet\lambda)+\rho_{\mathfrak{h}} = w(\lambda+\rho_{\mathfrak{g}})$ to the tensor product $V_\lambda\otimes\mathbb{S}_{\mathfrak{p}}\otimes\mathbb{S}_{\mathfrak{h}/\mathfrak{t}} \cong V_\lambda\otimes\mathbb{S}_{\mathfrak{g}/\mathfrak{t}}$. On the other hand, we see from the homogeneous Weyl formula (12) that the irreducible representations $U_{c\bullet\lambda}$ for $c\in C$ appear at least once in the decomposition of the tensor product $V_\lambda\otimes\mathbb{S}_{\mathfrak{g}/\mathfrak{h}}$. We therefore conclude that the representations $U_{c\bullet\lambda}$ for $c\in C$ each occur exactly once in $V_\lambda\otimes\mathbb{S}_{\mathfrak{p}}$. □

Lemma 10. *If μ is a weight of $V_\lambda\otimes\mathbb{S}_{\mathfrak{p}}$ satisfying $\|\mu+\rho_{\mathfrak{h}}\|^2 = \|\lambda+\rho_{\mathfrak{g}}\|^2$, then there exists a unique Weyl element $w\in W_{\mathfrak{g}}$ such that $\mu+\rho_{\mathfrak{h}} = w(\lambda+\rho_{\mathfrak{g}})$.*

Proof. If μ is a weight of $V_\lambda\otimes\mathbb{S}_{\mathfrak{p}}$, then $\mu+\rho_{\mathfrak{h}}$ is a weight of the tensor product $V_\lambda\otimes\mathbb{S}_{\mathfrak{p}}\otimes\mathbb{S}_{\mathfrak{h}/\mathfrak{t}} \cong V_\lambda\otimes\mathbb{S}_{\mathfrak{g}/\mathfrak{t}}$. Since the Weyl group acts simply transitively on the Weyl chambers, there exists an element $w\in W_{\mathfrak{g}}$ such that $w^{-1}(\mu+\rho_{\mathfrak{h}})$ is dominant, where we recall that a weight ν is dominant if and only if $\langle\nu,\alpha\rangle\geq 0$ for all positive roots α. Note that every weight of $\mathbb{S}_{\mathfrak{g}/\mathfrak{t}}$ can be obtained from its highest weight $\rho_{\mathfrak{g}}$

by subtracting a sum of positive roots. Likewise, for the tensor product $V_\lambda \otimes \mathbb{S}_{\mathfrak{g}/\mathfrak{t}}$, the difference $(\lambda + \rho_\mathfrak{g}) - w^{-1}(\mu + \rho_\mathfrak{h})$ is a sum of positive roots, and it follows that

$$\|\lambda + \rho_\mathfrak{g}\|^2 \geq \|w^{-1}(\mu + \rho_\mathfrak{h})\|^2,$$

with equality holding only when $(\lambda + \rho_\mathfrak{g}) - w^{-1}(\mu + \rho_\mathfrak{h}) = 0$. As for the uniqueness of w, if λ is dominant, then the weight $\lambda + \rho_\mathfrak{g}$ lies in the interior of the positive Weyl chamber for $\mathfrak{g}$, and thus the weights $w(\lambda + \rho_\mathfrak{g})$ for $w \in W_\mathfrak{g}$ are distinct. □

Theorem 8 now follows immediately from the above two lemmas. We can actually be slightly more specific about the kernel of the Dirac operator, recovering the signs appearing in the homogeneous Weyl formula (12). Recall that the spin representation decomposes as $\mathbb{S}_\mathfrak{p} = \mathbb{S}_\mathfrak{p}^+ \oplus \mathbb{S}_\mathfrak{p}^-$ into two half-spin representations. Since the Dirac operator is an odd element of the non-abelian Weil algebra $U(\mathfrak{g}) \otimes \mathrm{Cl}(\mathfrak{g})$, it interchanges $\mathbb{S}_\mathfrak{p}^+$ and $\mathbb{S}_\mathfrak{p}^-$. Restricting the domain of the Dirac operator to the positive half-spin representation, we obtain an operator

$$\not{\partial}^+_{\mathfrak{g}/\mathfrak{h}} : V_\lambda \otimes \mathbb{S}_\mathfrak{p}^+ \to V_\lambda \otimes \mathbb{S}_\mathfrak{p}^-.$$

Furthermore, since the Dirac operator is formally self-adjoint, its adjoint is

$$\not{\partial}^-_{\mathfrak{g}/\mathfrak{h}} : V_\lambda \otimes \mathbb{S}_\mathfrak{p}^- \to V_\lambda \otimes \mathbb{S}_\mathfrak{p}^+,$$

the restriction of the Dirac operator to the negative half-spin representation. Since these Dirac operator are acting on finite dimensional vector spaces, the index is the difference of the domain and range, so we have

$$\mathrm{Ker}\,\not{\partial}^+_{\mathfrak{g}/\mathfrak{h}} - \mathrm{Ker}\,\not{\partial}^-_{\mathfrak{g}/\mathfrak{h}} = V_\lambda \otimes \mathbb{S}_\mathfrak{p}^+ - V_\lambda \otimes \mathbb{S}_\mathfrak{p}^-, \tag{30}$$

which is given by the homogeneous Weyl formula (12). Comparing this with the kernel of $\not{\partial}_{\mathfrak{g}/\mathfrak{h}} = \not{\partial}^+_{\mathfrak{g}/\mathfrak{h}} \oplus \not{\partial}^-_{\mathfrak{g}/\mathfrak{h}}$ given in Theorem 8, we therefore obtain

$$\mathrm{Ker}\,\not{\partial}^+_{\mathfrak{g}/\mathfrak{h}} = \bigoplus_{(-1)^c = +1} U_{c\bullet\lambda}, \qquad \mathrm{Ker}\,\not{\partial}^-_{\mathfrak{g}/\mathfrak{h}} = \bigoplus_{(-1)^c = -1} U_{c\bullet\lambda}.$$

In other words, there is no cancellation on the left hand side of equation (30), and the signed kernel of this Dirac operator picks out precisely those representations, with sign, appearing on the right hand side of the homogeneous Weyl formula (12).

7. The Clifford algebra $\mathrm{Cl}(L\mathfrak{g})$

In Section 4, we examined the Clifford algebra associated to a finite dimensional Lie algebra with an invariant inner product. The infinite dimensional case is more complicated, and the general theory of such infinite dimensional Clifford algebras and their spin representations is developed in the mathematical literature by Kostant and Sternberg in [5]. Here, we consider the Clifford algebra associated to the Lie algebra $L\mathfrak{g}$ of smooth maps from S^1 to a finite dimensional Lie algebra $\mathfrak{g}$, where we restrict to the dense subspace of loops with finite Fourier expansions. This finiteness condition ensures that the Lie algebra $L\mathfrak{g}$ has a countable basis, and the complexification of this loop space is then $L\mathfrak{g}_\mathbb{C} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_\mathbb{C} z^k = \mathfrak{g}_\mathbb{C}[z, z^{-1}]$, the Lie algebra of finite Laurent series with values in $\mathfrak{g}_\mathbb{C}$. Averaging the pointwise inner products over the loop, we obtain an invariant inner product on $L\mathfrak{g}$ given by (3).

The Clifford algebra $\mathrm{Cl}(L\mathfrak{g})$ is spanned by *finite* sums of products of the form $\xi_1 \cdots \xi_n$ for loops $\xi_i \in L\mathfrak{g}$, subject to the relation $\{\xi, \eta\} = 2\langle \xi, \eta \rangle$. However, the loop space analogues of the elements $\widetilde{\mathrm{ad}}\, X$ and γ introduced in §4 are in fact infinite sums, so we must instead work with a formal completion of the Clifford algebra.

Unfortunately, the product of two such infinite sums does not necessarily converge. On the other hand, given a spin representation $\mathcal{S}_{L\mathfrak{g}}$ of the Clifford algebra $\mathrm{Cl}(L\mathfrak{g})$, we can view $\mathrm{End}(\mathcal{S}_{L\mathfrak{g}})$ as a completion of $\mathrm{Cl}(L\mathfrak{g})$ with a well defined product given by the composition of endomorphisms. As we discussed in §2, to define the spin representation we must first choose a polarization. With respect to the action of the infinitesimal generator ∂_θ of rotations, the complexified loop space $L\mathfrak{g}_\mathbb{C}$ decomposes into its negative, zero, and positive energy subspaces $L\mathfrak{g}_\mathbb{C} = L\mathfrak{g}_\mathbb{C}^- \oplus \mathfrak{g}_\mathbb{C} \oplus L\mathfrak{g}_\mathbb{C}^+$, where $L\mathfrak{g}_\mathbb{C}^+$ and $L\mathfrak{g}_\mathbb{C}^-$ are isotropic subspaces which are dual to each other with respect to the inner product. The spin representation corresponding to this polarization is $\mathcal{S}_{L\mathfrak{g}} := \mathbb{S}_\mathfrak{g} \otimes \Lambda^*(L\mathfrak{g}_\mathbb{C}^+)$, and the Clifford action $c : \mathrm{Cl}(L\mathfrak{g}_\mathbb{C}) \to \mathrm{End}(\mathcal{S}_{L\mathfrak{g}})$ is given by

$$c(\xi) = \begin{cases} 1 \otimes \varepsilon(\xi) & \text{for } \xi \in L\mathfrak{g}_\mathbb{C}^+, \\ 1 \otimes \iota(\xi) & \text{for } \xi \in L\mathfrak{g}_\mathbb{C}^-, \\ c(\xi) \otimes (-1)^F & \text{for } \xi \in \mathfrak{g}_\mathbb{C}, \end{cases}$$

where ε and ι are exterior multiplication and interior contraction respectively, and F is the degree operator on the exterior algebra. If $\{\eta_i\}$ is a basis for $\mathrm{Cl}(L\mathfrak{g}_\mathbb{C}^-)$, then when applied to a specific element of the spin representation $\mathcal{S}_{L\mathfrak{g}}$, all but finitely many of the operators $c(\eta_i) = \iota(\eta_i)$ vanish. Formal infinite sums $\sum_i c(\omega_i)c(\eta_i)$ with coefficients $\omega_i \in \mathrm{Cl}(\mathfrak{g}_\mathbb{C} \oplus L\mathfrak{g}_\mathbb{C}^+)$ therefore yield well defined operators on the spin representation, and in fact all elements of $\mathrm{End}(\mathcal{S}_{L\mathfrak{g}})$ can be expressed in this form.

The exterior algebra that we shall consider here is not $\Lambda^*(L\mathfrak{g}^*)$, but rather the algebra $\Lambda^*(L\mathfrak{g})^*$ of skew-symmetric multilinear forms on $L\mathfrak{g}$. Such forms can be expressed as formal infinite sums of basic products of the form $\xi_1^* \wedge \cdots \wedge \xi_n^*$ for $\xi_i \in L\mathfrak{g}$. In infinite dimensions, the Chevalley map $\mathrm{ch} : \mathrm{Cl}(L\mathfrak{g}) \to \Lambda^*(L\mathfrak{g})^*$ is no longer surjective; its image consists of all forms given by finite sums of the basic wedge products. Although the Chevalley map fails to converge if we attempt to extend it to the completion $\mathrm{End}(\mathcal{S}_{L\mathfrak{g}})$ of $\mathrm{Cl}(L\mathfrak{g})$, we can perturb it by terms of lower degree to remove the infinite contributions. Separating the Clifford algebra into its positive and negative energy factors, we define the *normal ordering* map $n : \mathrm{Cl}(L\mathfrak{g}_\mathbb{C}) \to \Lambda^*(L\mathfrak{g}_\mathbb{C})^*$ by

$$n(\omega^+ \cdot \omega^-) = \mathrm{ch}(\omega^+) \wedge \mathrm{ch}(\omega^-),$$

where $\omega^+ \in \mathrm{Cl}(\mathfrak{g}_\mathbb{C} \oplus L\mathfrak{g}_\mathbb{C}^+)$ and $\omega^- \in \mathrm{Cl}(L\mathfrak{g}_\mathbb{C}^-)$. The normal ordering map extends to the completion $\mathrm{End}(\mathcal{S}_{L\mathfrak{g}})$ of the Clifford algebra, and its image is the subspace $\Lambda^*(L\mathfrak{g}_\mathbb{C})^+ \subset \Lambda^*(L\mathfrak{g}_\mathbb{C})^*$ given by

$$\Lambda^*(L\mathfrak{g}_\mathbb{C})^+ = \left\{\omega \in \Lambda^*(L\mathfrak{g}_\mathbb{C})^* \;\middle|\; (\iota_\eta \omega)^+ \in \Lambda^*(L\mathfrak{g}_\mathbb{C}^-) \text{ for all } \eta \in \Lambda^*(L\mathfrak{g}_\mathbb{C}^-)\right\},$$

where $(\,)^+$ denotes the projection of $\Lambda^*(L\mathfrak{g}_\mathbb{C})^*$ onto $\Lambda^*(L\mathfrak{g}_\mathbb{C}^+)^*$, and we identify $\Lambda^*(L\mathfrak{g}_\mathbb{C}^-)$ with a subspace of $\Lambda^*(L\mathfrak{g}_\mathbb{C}^+)^*$ via the inner product. In terms of a basis $\{\eta_i\}$ for $\Lambda^*(L\mathfrak{g}^-)$, we may write elements of $\Lambda^*(L\mathfrak{g}_\mathbb{C})^+$ as formal infinite sums $\sum_i \omega_i^* \wedge \eta_i^*$, with $\omega_i \in \Lambda^*(\mathfrak{g}_\mathbb{C} \oplus L\mathfrak{g}_\mathbb{C}^+)$ living in the zero and positive energy components.

Remark. Decomposing $L\mathfrak{g}_\mathbb{C} = \bigoplus_{k\in\mathbb{Z}} \mathfrak{g}_\mathbb{C} z^k$ in terms of its energy grading, we define a secondary degree on $L\mathfrak{g}_\mathbb{C}$ counting only the negative energy contribution

$$\mathrm{sdeg}\, Xz^k = \begin{cases} 0 & \text{for } k \geq 0, \\ k & \text{for } k < 0, \end{cases}$$

where $X \in \mathfrak{g}_\mathbb{C}$ and Xz^k is the loop $z \mapsto Xz^k$ for $|z| = 1$. Let $L\mathfrak{g}_\mathbb{C}^* = \mathfrak{g}_\mathbb{C}^*[z, z^{-1}]$ denote the reduced dual of $L\mathfrak{g}_\mathbb{C}$. Extending sdeg to the exterior algebra $\Lambda^*(L\mathfrak{g}_\mathbb{C}^*)$,

we note that $\Lambda^*(L\mathfrak{g}_\mathbb{C})^+$ is the completion of $\Lambda^*(L\mathfrak{g}_\mathbb{C}^*)$ with respect to sdeg. In other words, $\Lambda^*(L\mathfrak{g}_\mathbb{C})^+$ consists of all formal infinite sums $\sum_i \omega_i$ of sdeg-homogeneous elements $\omega_i \in \Lambda^*(L\mathfrak{g}_\mathbb{C}^*)$ with $\operatorname{sdeg}\omega_i \to \infty$.

We can now use the normal ordering identification $n : \operatorname{End}(\mathcal{S}_{L\mathfrak{g}}) \to \Lambda^*(L\mathfrak{g}_\mathbb{C})^+$ to define product and bracket structures on $\Lambda^*(L\mathfrak{g}_\mathbb{C})^+$. The normal ordered product $\omega_1 \cdot_n \omega_2 = n(n^{-1}\omega_1 \cdot n^{-1}\omega_2)$ on the exterior algebra differs from the product induced by the Chevalley identification by terms of lower degree. However, many of the supercommutators remain unchanged. In particular, the normal ordered bracket with the dual $\xi^* \in L\mathfrak{g}_\mathbb{C}^* \cong \Lambda^1(L\mathfrak{g}_\mathbb{C})^+$ of a loop $\xi \in L\mathfrak{g}_\mathbb{C}$ is still given by

$$\begin{aligned}[]
[\xi^*, \omega^+ \wedge \omega^-]_n &= n\big[n^{-1}(\xi^*), n^{-1}(\omega^+ \wedge \omega^-)\big] = n[\xi, \operatorname{ch}^{-1}\omega^+ \cdot \operatorname{ch}^{-1}\omega^-] \\
&= n\big([\xi, \operatorname{ch}^{-1}\omega^+] \cdot \operatorname{ch}^{-1}\omega^- \pm \operatorname{ch}^{-1}\omega^+ \cdot [\xi, \operatorname{ch}^{-1}\omega^-]\big) \\
&= 2\iota_\xi \omega^+ \wedge \omega^- \pm \omega^+ \wedge 2\iota_\xi \omega^- = 2\,\iota_\xi(\omega^+ \wedge \omega^-),
\end{aligned}$$

for $\omega^+ \in \Lambda^*(\mathfrak{g}_\mathbb{C} \oplus L\mathfrak{g}_\mathbb{C}^+)^*$ and $\omega^- \in \Lambda^*(L\mathfrak{g}_\mathbb{C}^-)^*$ of homogeneous degree. Thus,

$$[\xi^*, \omega]_n = 2\,\iota_\xi\,\omega \text{ for } \xi \in L\mathfrak{g}_\mathbb{C} \text{ and } \omega \in \Lambda^*(L\mathfrak{g}_\mathbb{C})^+. \tag{31}$$

Reprising the discussion of §4, for any $\xi \in L\mathfrak{g}$, consider the 2-form $d\xi^*$ given by $d\xi^*(\eta, \zeta) = -\frac{1}{2}\langle \xi, [\eta, \zeta]\rangle$ for all $\eta, \zeta \in L\mathfrak{g}$. Although ∂_θ is not an element of $L\mathfrak{g}$, we can nevertheless define an analogous 2-form $d\partial_\theta^*$ by $d\partial_\theta^*(\xi, \eta) := \frac{1}{2}\langle \xi, \partial_\theta \eta\rangle$. Note that $d\partial_\theta^*$ is closed but not exact, so it defines a cohomology element in $H^2(L\mathfrak{g})$. Finally, the fundamental 3-form Ω is given by $\Omega(\xi, \eta, \zeta) = -\frac{1}{6}\langle \xi, [\eta, \zeta]\rangle$ for $\xi, \eta, \zeta \in L\mathfrak{g}$. These elements all lie in $\Lambda^*(L\mathfrak{g}_\mathbb{C})^+$, and they satisfy the identities

$$\iota_\xi\, d\eta^* = [\xi, \eta]^*, \qquad \iota_\xi\, d\partial_\theta^* = -(\partial_\theta \xi)^*, \qquad \iota_\xi\, \Omega = d\xi^*. \tag{32}$$

Using the normal ordered product and bracket on $\Lambda^*(L\mathfrak{g}_\mathbb{C})^+$ coming from $\operatorname{End}(\mathcal{S}_{L\mathfrak{g}})$, we obtain the loop space version of Corollary 6.

Theorem 11. *If $\mathfrak{g}$ is simple, then the elements 1, ξ^* for $\xi \in L\mathfrak{g}$, $\widetilde{\operatorname{ad}}\,\xi = \frac{1}{2}\,d\xi^*$ for $\xi \in \mathbb{R} \mathbin{\tilde{\oplus}} L\mathfrak{g}$, and $\gamma = \frac{1}{4}\,\Omega$ span a Lie superalgebra in $\Lambda^*(L\mathfrak{g})^+ \subset \operatorname{End}(\mathcal{S}_{L\mathfrak{g}})$ satisfying*

$$\begin{aligned}
\{\xi^*, \eta^*\} &= 2\langle \xi, \eta\rangle, & & \\
[\widetilde{\operatorname{ad}}\,\xi, \eta^*] &= [\xi, \eta]^*, & [\widetilde{\operatorname{ad}}\,\xi, \widetilde{\operatorname{ad}}\,\eta] &= \widetilde{\operatorname{ad}}\,[\xi, \eta] + ic_\mathfrak{g}\langle \xi, \partial_\theta \eta\rangle, \\
[\widetilde{\operatorname{ad}}\,\partial_\theta, \xi^*] &= (\partial_\theta \xi)^*, & [\widetilde{\operatorname{ad}}\,\partial_\theta, \widetilde{\operatorname{ad}}\,\xi] &= \widetilde{\operatorname{ad}}(\partial_\theta \xi), \\
\{\gamma, \xi^*\} &= \widetilde{\operatorname{ad}}\,\xi, & [\gamma, \widetilde{\operatorname{ad}}\,\xi] &= \tfrac{1}{2}\,ic_\mathfrak{g}(\partial_\theta \xi)^*, \\
[\gamma, \widetilde{\operatorname{ad}}\,\partial_\theta] &= 0, & \{\gamma, \gamma\} &= ic_\mathfrak{g}\,\widetilde{\operatorname{ad}}\,\partial_\theta - \tfrac{1}{24}\,c_\mathfrak{g} \dim \mathfrak{g},
\end{aligned}$$

where $c_\mathfrak{g}$ is the value of the Casimir operator of $\mathfrak{g}$ in the adjoint representation.

Proof. The bracket $\{\xi^*, \eta^*\} = 2\langle \xi, \eta\rangle$ is simply the definition of the Clifford algebra, while the brackets $[\widetilde{\operatorname{ad}}\,\xi, \eta^*] = [\xi, \eta]^*$ and $[\widetilde{\operatorname{ad}}\,\partial_\theta, \xi^*] = (\partial_\theta \xi)^*$ and $\{\gamma, \xi^*\} = \widetilde{\operatorname{ad}}\,\xi$ follow immediately from (31) and (32). By the Jacobi identity, for any $\xi, \eta \in \mathbb{R} \mathbin{\tilde{\oplus}} L\mathfrak{g}$ and $\zeta \in L\mathfrak{g}$ we have

$$\begin{aligned}
\big[[\widetilde{\operatorname{ad}}\,\xi, \widetilde{\operatorname{ad}}\,\eta], \zeta^*\big] &= \big[\widetilde{\operatorname{ad}}\,\xi, [\widetilde{\operatorname{ad}}\,\eta, \zeta^*]\big] - \big[\widetilde{\operatorname{ad}}\,\eta, [\widetilde{\operatorname{ad}}\,\xi, \zeta^*]\big] \\
&= \big[\xi, [\eta, \zeta]\big]^* - \big[\eta, [\xi, \zeta]\big]^* = \big[[\xi, \eta], \zeta\big]^* = \big[\widetilde{\operatorname{ad}}\,[\xi, \eta], \zeta^*\big],
\end{aligned}$$

which shows that $\widetilde{\operatorname{ad}}$ is a projective representation of $\mathbb{R} \mathbin{\tilde{\oplus}} L\mathfrak{g}$ on $\mathcal{S}_{L\mathfrak{g}}$. In Theorem 1, we established that this spin representation has central charge $c_\mathfrak{g}$, which gives us the brackets $[\widetilde{\operatorname{ad}}\,\xi, \widetilde{\operatorname{ad}}\,\eta] = \widetilde{\operatorname{ad}}\,[\xi, \eta] + ic_\mathfrak{g}\langle \xi, \partial_\theta \eta\rangle$ and $[\widetilde{\operatorname{ad}}\,\partial_\theta, \widetilde{\operatorname{ad}}\,\xi] = \widetilde{\operatorname{ad}}\,\partial_\theta \xi$.

To compute γ^2, we write it as the sum $\gamma^2 = (\gamma^2)_0 + (\gamma^2)_2$ of homogeneous forms of degrees 0 and 2. (We shall see that γ^2 has no components of degrees 4 or 6.) Since $\iota_\xi = \operatorname{ad}\xi^*$ for $\xi \in L\mathfrak{g}$ is a derivation with respect to the backet, we have

$$\iota_\xi \gamma^2 = [\iota_\xi\gamma, \gamma] = \tfrac{1}{2}\,[\widetilde{\operatorname{ad}}\,\xi, \gamma].$$

Taking one further interior contraction, we obtain

$$\iota_\xi\iota_\eta\gamma^2 = -\tfrac{1}{4}\big([\widetilde{\operatorname{ad}}\,\xi, \widetilde{\operatorname{ad}}\,\eta] - \widetilde{\operatorname{ad}}\,[\xi,\eta]\big) = -\tfrac{1}{4}\,ic_{\mathfrak{g}}\langle \xi, \partial_\theta\eta\rangle,$$

which is a constant. It follows that γ^2 has no components of degree higher than 2, and that $(\gamma^2)_2$ is the 2-cocycle determining the central extension of $L\mathfrak{g}$ for the spin representation $\widetilde{\operatorname{ad}}$. In fact, this 2-cocycle is a multiple of $\widetilde{\operatorname{ad}}\,\partial_\theta$, and we have

$$(\gamma^2)_2(\xi,\eta) = -\tfrac{1}{2}\,\iota_\xi\iota_\eta\gamma^2 = \tfrac{1}{8}\big(ic_{\mathfrak{g}}\langle\xi,\partial_\theta\eta\rangle\big) = \tfrac{1}{2}\,ic_{\mathfrak{g}}(\widetilde{\operatorname{ad}}\,\partial_\theta)(\xi,\eta).$$

Going back up one level, we see that

$$[\widetilde{\operatorname{ad}}\,\xi,\gamma] = 2\,\iota_\xi\gamma^2 = ic_{\mathfrak{g}}\,\iota_\xi\,\widetilde{\operatorname{ad}}\,\partial_\theta = -\tfrac{1}{2}\,ic_{\mathfrak{g}}(\partial_\theta\xi)^*.$$

Finally, the value of the constant $(\gamma^2)_0$ is the value of γ^2 acting on the minimum energy subspace $\mathcal{S}_{L\mathfrak{g}}(0)$ of the spin representation, since $\widetilde{\operatorname{ad}}\,\partial_\theta$ vanishes there. However, all the terms in γ^2 vanish on $\mathcal{S}_{L\mathfrak{g}}(0)$ except the contribution from the constant loops, and thus $(\gamma^2)_0 = \frac{1}{48}\operatorname{tr}_{\mathfrak{g}}\Delta^{\mathfrak{g}}_{\mathrm{ad}} = \frac{1}{48}\,c_{\mathfrak{g}}\dim\mathfrak{g}$ as we proved in Corollary 6. □

Taking a slightly different view of this theorem, the commutation relations given in Theorem 11 determine a Lie superalgebra (with subscripts denoting the grading)

$$\mathbb{R}_{\mathrm{even}} \oplus L\mathfrak{g}_{\mathrm{odd}} \oplus (\mathbb{R}\,\tilde{\oplus}\,L\mathfrak{g})_{\mathrm{even}} \oplus \mathbb{R}_{\mathrm{odd}},$$

and the identification of $\Lambda^*(L\mathfrak{g})^+$ with its image in $\operatorname{End}(\mathcal{S}_{L\mathfrak{g}})$ gives a representation of this Lie superalgebra on the spin representation $\mathcal{S}_{L\mathfrak{g}}$. Actually, we can extend this Lie superalgebra further. The component $\mathbb{R}_{\mathrm{even}} \oplus L\mathfrak{g}_{\mathrm{odd}} \oplus L\mathfrak{g}_{\mathrm{even}}$ is called a *super Kac-Moody algebra*, and using superspace notation, its complexification is a central extension of the polynomial algebra $\mathfrak{g}\otimes\mathbb{C}[z,z^{-1},\Theta]$, where Θ is an odd variable (i.e. $\Theta^2=0$). The *super Virasoro algebra* SVir is the universal central extension of the Lie algebra of derivations of $\mathbb{C}[z,z^{-1},\Theta]$. (Note that the even derivations are just the vector fields on the circle.) The super Virasoro algebra therefore acts on the super Kac-Moody algebra, and their semidirect sum is referred to as the $N=1$ *superconformal current algebra* (see [8]):

$$\mathrm{SVir}\;\tilde{\oplus}\;(\mathbb{R}_{\mathrm{even}} \oplus L\mathfrak{g}_{\mathrm{odd}} \oplus L\mathfrak{g}_{\mathrm{even}}).$$

In our case, the elements $\widetilde{\operatorname{ad}}\,\partial_\theta$ and γ span the even and odd zero-mode subspaces of the super Virasoro algebra, with commutator $\{\gamma,\gamma\} = ic_{\mathfrak{g}}\big(\widetilde{\operatorname{ad}}\,\partial_\theta + \frac{1}{24}\,i\dim\mathfrak{g}\big)$. Here, the additional $\frac{1}{24}\dim\mathfrak{g}$ term, which is sometimes incorporated into the definition of $\widetilde{\operatorname{ad}}\,\partial_\theta$, corresponds to the anomalous energy shift we encountered in (9).

Given an orthonormal basis $\{X_i\}$ for $\mathfrak{g}$, the loops $X_i^n = X_i z^n$ for $n\in\mathbb{Z}$ form a basis for $L\mathfrak{g}_{\mathbb{C}}$ satisfying $\langle X_i^n, X_j^m\rangle = \delta_{i,j}\delta_{n,-m}$. In terms of this basis, we have

$$\widetilde{\operatorname{ad}}\,\xi = -\frac{1}{4}\sum_{i,k} X_i^{-k}\cdot[\xi, X_i^k], \qquad \widetilde{\operatorname{ad}}\,\partial_\theta = \frac{1}{2}\sum_{j,\,k>0} ik\,X_j^k\cdot X_j^{-k},$$

$$\gamma = -\frac{1}{24}\sum_{i,j,k,l} X_i^{-k}\cdot X_j^{-l}\cdot[X_i,X_j]^{k+l} = \frac{1}{6}\sum_{i,k} X_i^{-k}\cdot\widetilde{\operatorname{ad}}\,X_i^k.$$

Note that in the expressions for $\widetilde{\operatorname{ad}}\,\xi$ and γ, the ordering of the factors does not matter (up to sign), since they are orthogonal and therefore anti-commute with

each other. However, in the expression for $\widetilde{\mathrm{ad}}\,\partial_\theta$, we have $\{X_i^k, X_i^{-k}\} = 2$, so changing the order of the factors shifts the operator by a constant. Here we see normal ordering in action, forcing us to write factors X_i^k with k positive on the left and factors X_i^{-k} with $-k$ negative on the right. In physics notation, this would be written as $\widetilde{\mathrm{ad}}\,\partial_\theta = -\frac{1}{4}\sum_{j,\,k\in\mathbb{Z}} ik : X_j^{-k}\,X_j^k :$, where $:\xi\,\eta: = n^{-1}(n\,\xi\wedge n\,\eta)$ denotes the normal ordered product in the Clifford algebra. (This colon notation is misleading as it is not a map on the Clifford algebra but rather an instruction to replace all Clifford products between the colons with normal ordered products.)

If the Lie algebra $\mathfrak{g}$ is not simple, then Theorem 11 still holds, albeit with slightly modified commutation relations. For a general finite dimensional Lie algebra $\mathfrak{g}$, the Casimir operator $\Delta^{\mathfrak{g}}_{\mathrm{ad}} = -\frac{1}{2}\sum_i (\mathrm{ad}\,X_i)^2$ no longer takes a constant value $c_{\mathfrak{g}}$. In this case, the role of the quadratic element $\widetilde{\mathrm{ad}}\,\partial_\theta$ is played by the 2-cocycle $\omega_{\widetilde{\mathrm{ad}}}$ for the projective spin representation $\widetilde{\mathrm{ad}}$, given on $\xi, \eta \in L\mathfrak{g}$ by

$$\omega_{\widetilde{\mathrm{ad}}}(\xi,\eta) := [\widetilde{\mathrm{ad}}\,\xi, \widetilde{\mathrm{ad}}\,\eta] - \widetilde{\mathrm{ad}}\,[\xi,\eta] = i\,\langle \xi, \Delta^{\mathfrak{g}}_{\mathrm{ad}}\partial_\theta\eta\rangle,$$

where the Casimir operator $\Delta^{\mathfrak{g}}_{\mathrm{ad}}$ acts pointwise on the loop space $L\mathfrak{g}$. Viewing $\omega_{\widetilde{\mathrm{ad}}}$ as an element of the Clifford algebra, we have the commutator

$$[\omega_{\widetilde{\mathrm{ad}}}, \xi^*] = -2\,\iota_\xi\,\omega_{\widetilde{\mathrm{ad}}} = 4i\left(\Delta^{\mathfrak{g}}_{\mathrm{ad}}\partial_\theta\xi\right)^*,$$

so we may also view the projective cocycle as $\omega_{\widetilde{\mathrm{ad}}} = 4i\,\widetilde{\mathrm{ad}}(\Delta^{\mathfrak{g}}\partial_\theta)$, where $\Delta^{\mathfrak{g}}$ is the formal Casimir operator in the universal enveloping algebra of $\mathfrak{g}$. We therefore have

$$[\omega_{\widetilde{\mathrm{ad}}}, \widetilde{\mathrm{ad}}\,\xi] = 4i\,[\widetilde{\mathrm{ad}}(\Delta^{\mathfrak{g}}\partial_\theta), \widetilde{\mathrm{ad}}\,\xi] = 4i\,\widetilde{\mathrm{ad}}\big(\Delta^{\mathfrak{g}}_{\mathrm{ad}}\partial_\theta\xi\big),$$

and the adjoint action of γ in Theorem 11 then becomes

$$\begin{aligned} [\gamma, \widetilde{\mathrm{ad}}\,\xi] &= \tfrac{1}{2}\,i\left(\Delta^{\mathfrak{g}}_{\mathrm{ad}}\partial_\theta\xi\right)^*, \\ \{\gamma,\gamma\} &= \tfrac{1}{4}\,\omega_{\widetilde{\mathrm{ad}}} - \tfrac{1}{24}\,\mathrm{tr}_{\mathfrak{g}}\,\Delta^{\mathfrak{g}}_{\mathrm{ad}}, \end{aligned}$$

with the other commutation relations remaining unchanged. Alternatively, the projective cocycle $\omega_{\widetilde{\mathrm{ad}}}$ can be viewed as the 2-form component of the Casimir operator

$$\Delta^{L\mathfrak{g}}_{\widetilde{\mathrm{ad}}} = -2i\,\widetilde{\mathrm{ad}}(\Delta^{\mathfrak{g}}\partial_\theta) + \Delta^{\mathfrak{g}}_{\widetilde{\mathrm{ad}}} = -\tfrac{1}{2}\,\omega_{\widetilde{\mathrm{ad}}} + \tfrac{1}{8}\,\mathrm{tr}_{\mathfrak{g}}\,\Delta^{\mathfrak{g}}_{\mathrm{ad}}$$

for the spin representation $\widetilde{\mathrm{ad}}$ of $L\mathfrak{g}$, which we discuss in Theorem 12 below.

8. The Dirac operator on $L\mathfrak{g}$

Following our discussion in Section 5, given any positive energy representation $r : \tilde{L}\mathfrak{g} \to \mathrm{End}(\mathcal{H})$, we construct a Dirac operator

$$\not{\partial}_r := \hat{r} + 1 \otimes \tfrac{1}{2}\,\Omega_{L\mathfrak{g}} \in \mathrm{End}(\mathcal{H}\otimes\mathcal{S}_{L\mathfrak{g}}),$$

where $\hat{r}$ is the tautological $\mathrm{End}(\mathcal{H})$-valued 1-form on $L\mathfrak{g}$ given by $\hat{r}(\xi) = r(\xi)$ for all $\xi \in L\mathfrak{g}$, and $\Omega_{L\mathfrak{g}}$ is the fundamental 3-form given by $\Omega_{L\mathfrak{g}}(\xi,\eta,\zeta) = -\frac{1}{6}\langle\xi,[\eta,\zeta]\rangle$ for $\xi,\eta,\zeta \in L\mathfrak{g}$. As in the previous section, we implicitly identify $\Lambda^*(L\mathfrak{g})^+$ with its image in $\mathrm{End}(\mathcal{S}_{L\mathfrak{g}})$. Written in terms of a basis $\{X_i^n\}$ of $L\mathfrak{g}$ satisfying $\langle X_i^n, X_j^m\rangle = \delta_{i,j}\delta_{n,-m}$, this Dirac operator is

$$\begin{aligned} \not{\partial}_r &= \sum_{i,n} X_i^{-n}\,r(X_i^n) - \frac{1}{12}\sum_{i,j,m,n} X_i^{-n}\cdot X_j^{-m}\cdot [X_i,X_j]^{n+m} \\ &= \sum_{i,n} X_i^{-n}\Big(r(X_i^n) + \frac{1}{3}\,\widetilde{\mathrm{ad}}\,X_i^n\Big). \end{aligned}$$

Note that all of the individual factors in this expression (anti-)commute with each other, so $\partial\!\!\!/_r$ does indeed give a well-defined operator on the tensor product $\mathcal{H}\otimes\mathcal{S}_{L\mathfrak{g}}$, without requiring normal ordering or dealing with any infinite constants.

In its most general form, if we take the representation r to be the canonical inclusion $r : L\mathfrak{g} \hookrightarrow U(L\mathfrak{g})$ of $L\mathfrak{g}$ into its universal enveloping algebra $U(L\mathfrak{g})$, then the corresponding universal Dirac operator is an element of the formal completion of the non-abelian Weil algebra $\mathcal{A} = U(\tilde{L}\mathfrak{g})\otimes\mathrm{Cl}(L\mathfrak{g})$. (Alternatively, we may view $\mathcal{A}$ as the universal enveloping algebra of the super Kac-Moody algebra $\tilde{L}\mathfrak{g}_{\text{even}}\oplus L\mathfrak{g}_{\text{odd}}$.) As we saw in the previous section, the product of two such infinite formal sums does not necessarily converge. However, keeping in mind that we are really working with operators on Hilbert spaces, we can indeed extend multiplication to a suitable subspace $\mathcal{A}^+$ of the formal completion, which we define as the largest subspace for which the homomorphism $\mathcal{A} \to \mathrm{End}(\mathcal{H}\otimes\mathcal{S}_{L\mathfrak{g}})$ extends to $\mathcal{A}^+$ for any positive energy representation $\mathcal{H}$ of $\tilde{L}\mathfrak{g}$. In particular, if $\mathcal{H}$ is a faithful representation of $U(\tilde{L}\mathfrak{g})$—we can construct such a representation by taking the Hilbert space direct sum of countably many irreducible positive energy representations of $\tilde{L}\mathfrak{g}$—then the homomorphism $\mathcal{A}^+ \hookrightarrow \mathrm{End}(\mathcal{H}\otimes\mathcal{S}_{L\mathfrak{g}})$ induces a product structure on $\mathcal{A}^+$. Fortunately, we can perform all of our computations here using the techniques of the previous section, working with $U(\tilde{L}\mathfrak{g})$-valued forms on $\tilde{L}\mathfrak{g}$.

Using this extended multiplication, the square of the Dirac operator is

$$\partial\!\!\!/^2 = \hat{r}^2 + \{\hat{r}, \tfrac{1}{2}\,\Omega_{L\mathfrak{g}}\} + \tfrac{1}{4}\,\Omega_{L\mathfrak{g}}^2.$$

Since $\hat{r}$ is an $\mathrm{End}(\mathcal{H})$-valued 1-form on $L\mathfrak{g}$, its square is a sum $\hat{r}^2 = (\hat{r}^2)_0 + (\hat{r}^2)_2$ of forms of homogeneous degrees 0 and 2. For the degree 2 component, we have $(\hat{r}^2)_2 = \hat{r}\wedge\hat{r}$, and the "curvature" $d\hat{r} + \hat{r}\wedge\hat{r}$ of the representation r is given by

$$(d\hat{r} + \hat{r}\wedge\hat{r})(\xi,\eta) = \tfrac{1}{2}\big(\,[r(\xi), r(\eta)] - r([\xi,\eta])\,\big) = \tfrac{1}{2}\,\omega_r(\xi,\eta),$$

where $\omega_r \in \Lambda^2(L\mathfrak{g})^+$ is the 2-cocycle corresponding to the projective representation r. If $\mathfrak{g}$ is simple and I is the generator of the universal central extension of $L\mathfrak{g}$, then $\omega_r = 4\,r(I)\,\widetilde{\mathrm{ad}}\,\partial_\theta$. The degree 0 component of $\hat{r}^2$ is given by the following:

Theorem 12. *The operator $\Delta_r^{L\mathfrak{g}} := -\frac{1}{2}\big(\hat{r}^2\big)_0$ is called the Casimir operator for the loop group $L\mathfrak{g}$, and if $\mathfrak{g}$ is simple then the Casimir operator acting on the irreducible positive energy representation $\mathcal{H}_{\boldsymbol{\lambda}}$ with lowest weight $\boldsymbol{\lambda} = (m, -\lambda, h)$ is given by*

$$\begin{aligned}\Delta_r^{L\mathfrak{g}} &= -i\,(h + c_{\mathfrak{g}})\,(r(\partial_\theta) - im) + \Delta_\lambda^{\mathfrak{g}}\\ &= -i\,(h + c_{\mathfrak{g}})\,r(\partial_\theta) + \tfrac{1}{2}\big(\|\boldsymbol{\lambda} - \boldsymbol{\rho}_{\mathfrak{g}}\|^2 - \|\boldsymbol{\rho}_{\mathfrak{g}}\|^2\big),\end{aligned} \tag{33}$$

where $\boldsymbol{\rho}_{\mathfrak{g}} = (0, \rho_{\mathfrak{g}}, -c_{\mathfrak{g}})$ and $c_{\mathfrak{g}} = \Delta_{\mathrm{ad}}^{\mathfrak{g}}$ is the value of the quadratic Casimir operator of $\mathfrak{g}$ acting on the adjoint representation, and the inner product is given by (6).

Proof. In order to simply our calculations, we first note the following identites:

$$\begin{aligned}[\hat{r}, \widetilde{\mathrm{ad}}\,\xi](\eta) &= [r(\eta), \widetilde{\mathrm{ad}}\,\xi] - \{\hat{r}, \tfrac{1}{2}[\eta,\xi]^*\} = r([\xi,\eta]),\\ [\hat{r}, r(\xi)](\eta) &= [r(\eta), r(\xi)] = r([\eta,\xi]) + \langle \eta, \partial_\theta\xi\rangle\, r(I)\\ &= \big([\widetilde{\mathrm{ad}}\,\xi, \hat{r}] + r(I)(\partial_\theta\xi)^*\big)(\eta), \text{ and}\\ [\widetilde{\mathrm{ad}}\,\xi, \hat{r}^2]_0 &= -[\widetilde{\mathrm{ad}}\,\xi, (\hat{r}^2)_2]_0 = -[\widetilde{\mathrm{ad}}\,\xi, d\hat{r}]_0\\ &= 2\,(\widetilde{\mathrm{ad}}\,I)\,\hat{r}(\partial_\theta\xi) = 2\,(\widetilde{\mathrm{ad}}\,I)\,r(\partial_\theta\xi).\end{aligned}$$

We now show that the commutator of $\Delta_r^{L\mathfrak{g}}$ with an element $\xi \in L\mathfrak{g}$ is

$$\begin{aligned}[\Delta_r^{L\mathfrak{g}}, r(\xi)] &= -\tfrac{1}{2}\,[\hat{r}^2, r(\xi)]_0 = -\tfrac{1}{2}\,\big\{\hat{r}, [\hat{r}, r(\xi)]\big\}_0 \\ &= \tfrac{1}{2}\,\big\{\hat{r}, [\hat{r}, \widetilde{\mathrm{ad}}\,\xi]\big\}_0 - \tfrac{1}{2}\,\big\{\hat{r}, r(I)(\partial_\theta\xi)^*\big\}_0 \\ &= -(\widetilde{\mathrm{ad}}\,I)\,r(\partial_\theta\xi) - r(I)\,r(\partial_\theta\xi) \\ &= -\big[(r(I) + \widetilde{\mathrm{ad}}\,I)\,r(\partial_\theta), r(\xi)\big].\end{aligned}$$

It follows that the operator $\tilde{\Delta}_r^{L\mathfrak{g}} := \Delta_r^{L\mathfrak{g}} + (r(I) + \widetilde{\mathrm{ad}}\,I)\,r(\partial_\theta)$ commutes with the action of $L\mathfrak{g}$, and therefore takes a constant value on each irreducible representation. Acting on the minimum energy subspace $\mathcal{H}_{\boldsymbol{\lambda}}(m)$ of $\mathcal{H}_{\boldsymbol{\lambda}}$, the only terms contributing to $\Delta_r^{L\mathfrak{g}}$ are those coming from the constant loops, and thus this constant is

$$\tilde{\Delta}_{\boldsymbol{\lambda}}^{L\mathfrak{g}} = \Delta_r^{L\mathfrak{g}}\big|_{\mathcal{H}_{\boldsymbol{\lambda}}(m)} + i\,(h + c_g)\,r(\partial_\theta)\big|_{\mathcal{H}_{\boldsymbol{\lambda}}(m)} = \Delta_\lambda^{\mathfrak{g}} - (h + c_{\mathfrak{g}})\,m.$$

The desired result then follows immediately. □

By definition, the 0-form component of $\hat{r}^2$ acts as the identity operator on $\mathcal{S}_{L\mathfrak{g}}$. To compute the action of $\Delta_r^{L\mathfrak{g}}$, we can therefore restrict it to the minimum energy subspace $\mathcal{S}_{L\mathfrak{g}}(0)$ of the spin representation. In terms of a basis $\{X_i^n\}$, we have

$$\begin{aligned}\Delta_r^{L\mathfrak{g}} &= -\frac{1}{2}\sum\nolimits_{i,n} r(X_i^n)\,X_i^{-n}\sum\nolimits_{j,m} r(X_j^m)\,X_j^{-m}\Big|_{\mathcal{H}\otimes\mathcal{S}_{L\mathfrak{g}}(0)} \\ &= -\frac{1}{2}\sum\nolimits_{i,j}\Big(r(X_i)\,r(X_j)\,X_i\cdot X_j + \sum\nolimits_{n>0} r(X_i^n)\,r(X_j^{-n})\,X_i^{-n}\cdot X_j^n\Big) \\ &= \Delta_r^{\mathfrak{g}} - \sum\nolimits_{i,\,n>0} r(X_i^n)\,r(X_i^{-n}),\end{aligned}$$

which is the usual definition of the Casimir operator for a loop group. The Casimir operator can be used to define the energy operator $r(\partial_\theta)$ in terms of the action of $L\mathfrak{g}$. The constant term $\Delta_\lambda^{\mathfrak{g}}$ is sometimes incorporated into $r(\partial_\theta)$, in which case it is viewed as an anomalous energy shift due to the degeneracy of the vacuum.

Returning to our computation of the square of the Dirac operator, we note that the cross term is given by the anti-commutator $\{\hat{r}, \tfrac{1}{2}\,\Omega_{L\mathfrak{g}}\} = d\hat{r}$, and we obtain

$$\partial\!\!\!/^2 = -2\,\Delta_r^{L\mathfrak{g}} + \tfrac{1}{2}\,\omega_\varrho - \tfrac{1}{12}\,\mathrm{tr}_{\mathfrak{g}}\,\Delta_{\mathrm{ad}}^{\mathfrak{g}},$$

where ω_ϱ is the 2-cocycle corresponding to the diagonal action $\varrho = r\otimes 1 + 1\otimes\widetilde{\mathrm{ad}}$ on the tensor product $\mathcal{H}\otimes\mathcal{S}_{L\mathfrak{g}}$. If $\mathfrak{g}$ is simple, then this 2-cocycle is $\omega_\varrho = 4\,\varrho(I)\,\widetilde{\mathrm{ad}}\,\partial_\theta$. Furthermore, if $\mathcal{H}_{\boldsymbol{\lambda}}$ is the irreducible positive energy representation of $L\mathfrak{g}$ with lowest weight $\boldsymbol{\lambda} = (m, -\lambda, h)$, then using (33) for the Casimir operator, we have

$$\begin{aligned}\partial\!\!\!/_{\boldsymbol{\lambda}}^2 &= 2i\,(h + c_{\mathfrak{g}})\,\big(\varrho(\partial_\theta) - im\big) - \|\lambda + \rho_g\|^2 \\ &= 2\,\varrho(I)\,\varrho(\partial_\theta) - \|\boldsymbol{\lambda} - \boldsymbol{\rho}_{\mathfrak{g}}\|^2.\end{aligned} \tag{34}$$

Note that unlike the finite dimensional case discussed in §5, the square of the Dirac operator for $L\mathfrak{g}$ does not take a constant value on each irreducible representation. Here, the Dirac operator fails to commute with the diagonal action ϱ of $L\mathfrak{g}$ on $\mathcal{H}\otimes\mathcal{S}_{L\mathfrak{g}}$. Since the Dirac operator satisfies the identity $\iota_\xi\,\partial\!\!\!/ = \varrho(\xi)$, we have

$$[\varrho(\xi), \partial\!\!\!/] = \iota_\xi\,\partial\!\!\!/^2 = \tfrac{1}{2}\,\iota_\xi\,\omega_\varrho = 2\,\varrho(I)\,\iota_\xi\,\widetilde{\mathrm{ad}}\,\partial_\theta = -\varrho(I)(\partial_\theta\xi)^*,$$

and thus $\partial\!\!\!/$ commutes only with the subalgebra $\mathbb{R}\oplus\mathfrak{g}\oplus\mathbb{R}$ of $\mathbb{R}\,\tilde{\oplus}\,\tilde{L}\mathfrak{g}$.

If the Lie algebra $\mathfrak{g}$ is reductive but not simple, then the expression (34) for the square of the Dirac operator still holds provided that the central extension satisfies $\omega_\varrho = 4\,\varrho(I)\,\widetilde{\mathrm{ad}}\,\partial_\theta$. In other words, the invariant inner product on $\mathfrak{g}$ must satisfy

$$[\varrho(\xi), \varrho(\eta)] - \varrho([\xi, \eta]) = \varrho(I)\langle \xi, \partial_\theta \eta\rangle$$

for some imaginary constant $\varrho(I)$. Given any irreducible positive energy projective representation $\mathcal{H}$ of $L\mathfrak{g}$, we can always choose an invariant inner product on $\mathfrak{g}$ such that $\mathcal{H} \otimes \mathcal{S}_{L\mathfrak{g}}$ is a true representation of the corresponding central extension $\tilde{L}\mathfrak{g}$. However, this choice of inner product depends on the representation, so this approach does not give a universal expression for the Dirac operator.

9. The Dirac operator on $L\mathfrak{g}/L\mathfrak{h}$

As in Section 6, let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$, and let $\mathfrak{p}$ denote the orthogonal complement of $\mathfrak{h}$ with respect to the invariant inner product on $\mathfrak{g}$. This orthogonal decomposition extends to the loop Lie algebra $L\mathfrak{g} = L\mathfrak{h} \oplus L\mathfrak{p}$, and the Clifford algebra decomposes as $\mathrm{Cl}(L\mathfrak{g}) \cong \mathrm{Cl}(L\mathfrak{h}) \otimes \mathrm{Cl}(L\mathfrak{p})$. If $\mathfrak{p}$ is even dimensional, as is the case when $\mathfrak{h}$ has the same rank as $\mathfrak{g}$, then we can also factor the spin representation as $\mathcal{S}_{L\mathfrak{g}} \cong \mathcal{S}_{L\mathfrak{h}} \otimes \mathcal{S}_{L\mathfrak{p}}$, where $\mathcal{S}_{L\mathfrak{h}}$ and $\mathcal{S}_{L\mathfrak{p}}$ are representations of $\tilde{L}\mathfrak{h}$ of levels $c_\mathfrak{h}$ and $c_\mathfrak{g} - c_\mathfrak{h}$ respectively, and the action of $L\mathfrak{h}$ on $\mathcal{S}_{L\mathfrak{p}}$ is

$$\begin{aligned} \widetilde{\mathrm{ad}}_{L\mathfrak{p}} : L\mathfrak{h} &\to \Lambda^2(L\mathfrak{p})^+ \hookrightarrow \mathrm{End}(\mathcal{S}_{L\mathfrak{p}}) \\ \zeta &\mapsto (\widetilde{\mathrm{ad}}_{L\mathfrak{p}}\,\zeta)(\xi, \eta) = \tfrac{1}{4}\,\langle \xi, [\zeta, \eta]\rangle \end{aligned}$$

for $\zeta \in L\mathfrak{h}$ and $\xi, \eta \in L\mathfrak{p}$.

Given any positive energy representation $r_{L\mathfrak{g}}$ of $\tilde{L}\mathfrak{g}$ on a Hilbert space $\mathcal{H}$, its restriction gives a representation $r_{L\mathfrak{h}}$ of $\tilde{L}\mathfrak{h}$ on $\mathcal{H}$. Now consider the diagonal representation $r'_{L\mathfrak{h}} = r_{L\mathfrak{h}} \otimes 1 + 1 \otimes \widetilde{\mathrm{ad}}_{L\mathfrak{p}}$ of $\tilde{L}\mathfrak{h}$ on the tensor product $\mathcal{H} \otimes \mathcal{S}_{L\mathfrak{p}}$. Using the construction of the previous section, we build the twisted Dirac operator

$$\partial\!\!\!/'_{L\mathfrak{h}} = \hat{r}'_{L\mathfrak{h}} + \tfrac{1}{2}\,\Omega_{L\mathfrak{h}} \in \mathrm{End}(\mathcal{H} \otimes \mathcal{S}_{L\mathfrak{p}} \otimes \mathcal{S}_{L\mathfrak{h}}) \cong \mathrm{End}(\mathcal{H} \otimes \mathcal{S}_{L\mathfrak{g}}).$$

Noting that the diagonal action $\varrho'_{L\mathfrak{h}} = r' \otimes 1 + 1 \otimes \widetilde{\mathrm{ad}}_{L\mathfrak{h}}$ on $\mathcal{H} \otimes \mathcal{S}_{L\mathfrak{g}}$ is simply the restriction of the action $\varrho_{L\mathfrak{g}} = r \otimes 1 + 1 \otimes \widetilde{\mathrm{ad}}_{L\mathfrak{g}}$ to $L\mathfrak{h}$, we obtain the identities

$$\begin{aligned} \iota_\zeta\, \partial\!\!\!/'_{L\mathfrak{h}} &= \varrho'_{L\mathfrak{h}}(\zeta) = \varrho_{L\mathfrak{g}}(\zeta), \\ [\varrho_{L\mathfrak{g}}(\zeta), \partial\!\!\!/'_{L\mathfrak{h}}] &= \tfrac{1}{2}\,\iota_\zeta\,\omega^{L\mathfrak{h}}_{\varrho'} = \tfrac{1}{2}\,\iota_\zeta\,\omega^{L\mathfrak{g}}_{\varrho}, \end{aligned}$$

for $\zeta \in L\mathfrak{h}$. The difference $\partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}} := \partial\!\!\!/_{L\mathfrak{g}} - \partial\!\!\!/'_{L\mathfrak{h}}$ is basic with respect to $L\mathfrak{h}$, i.e.

$$\iota_\zeta\, \partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}} = 0, \qquad [\varrho_{L\mathfrak{g}}(\zeta), \partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}}] = 0,$$

for all $\zeta \in L\mathfrak{h}$, and thus it can be written as the $L\mathfrak{h}$-equivariant operator

$$\partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}} = \hat{r}_{L\mathfrak{p}} + \tfrac{1}{2}\,\Omega_{L\mathfrak{p}} \in \mathrm{End}(\mathcal{H} \otimes \mathcal{S}_{L\mathfrak{p}})^{L\mathfrak{h}}, \tag{35}$$

where $\hat{r}_{L\mathfrak{p}}$ is the tautological $\mathrm{End}(\mathcal{H})$-valued 1-form on $L\mathfrak{p}$ given by $\hat{r}(\xi) = r(\xi)$ for $\xi \in L\mathfrak{g}$, and $\Omega_{L\mathfrak{p}}$ is the fundamental 3-form given by $\Omega_{L\mathfrak{p}}(\xi, \eta, \zeta) = -\frac{1}{6}\langle \xi, [\eta, \zeta]\rangle$ for $\xi, \eta, \zeta \in L\mathfrak{p}$. Writing this Dirac operator in terms of a basis $\{X_i^n\}$ of $L\mathfrak{p}$ satisfying $\langle X_i^n, X_j^m\rangle = \delta_{i,j}\delta_{n,-m}$, we have

$$\partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}} = \sum_{i,n} X_i^{-n}\, r(X_i^n) - \frac{1}{12} \sum_{i,j,m,n} X_i^{-n} \cdot X_j^{-m} \cdot [X_i, X_j]_\mathfrak{p}^{n+m},$$

where $[X, Y]_\mathfrak{p}$ denotes the projection of $[X, Y]$ onto $\mathfrak{p}$.

As we saw in the finite dimensional case, the two Dirac operators $\partial\!\!\!/'_{L\mathfrak{h}}$ and $\partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}}$ are decoupled, or in other words they anti-commute with each other:

$$\big\{\partial\!\!\!/'_{L\mathfrak{h}}, \partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}}\big\} = \big\{\hat{r}'_{L\mathfrak{h}}, \partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}}\big\} + \big\{\tfrac{1}{2}\,\Omega_{L\mathfrak{h}}, \partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}}\big\} = 0,$$

where the first summand vanishes since for all $\zeta \in L\mathfrak{h}$ we have

$$\big\{\hat{r}'_{L\mathfrak{h}}, \partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}}\big\}(\zeta) = [r'(\zeta), \partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}}] = 0,$$

and the second summand vanishes as the odd operators $\frac{1}{2}\,\Omega_{L\mathfrak{h}}$ and $\partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}}$ act on distinct representations $\mathcal{S}_{L\mathfrak{h}}$ and $\mathcal{H}\otimes\mathcal{S}_{L\mathfrak{p}}$ and therefore anti-commute. Since these two operators are decoupled, the square of the Dirac operator on $L\mathfrak{g}/L\mathfrak{h}$ is

$$\begin{aligned}\partial\!\!\!/^2_{L\mathfrak{g}/L\mathfrak{h}} &= \big(\partial\!\!\!/_{L\mathfrak{g}}\big)^2 - \big(\partial\!\!\!/'_{L\mathfrak{h}}\big)^2\\ &= -2\,\big(\Delta_r^{L\mathfrak{g}} - \Delta_{r'}^{L\mathfrak{h}}\big) + \tfrac{1}{2}\,\big(\omega_\varrho^{L\mathfrak{g}} - \omega_{\varrho'}^{L\mathfrak{h}}\big) + \tfrac{1}{12}\,\big(\mathrm{tr}_\mathfrak{g}\,\Delta^\mathfrak{g}_{\mathrm{ad}} - \mathrm{tr}_\mathfrak{h}\,\Delta^\mathfrak{h}_{\mathrm{ad}}\big).\end{aligned}$$

Now consider the case where $\mathfrak{g}$ is simple, $\mathfrak{h}$ is reductive, and $\mathcal{H}_{\boldsymbol{\lambda}}$ is the irreducible positive energy representation of $L\mathfrak{g}$ with lowest weight $\boldsymbol{\lambda}$. Since $\partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}}$ is an $L\mathfrak{h}$-equivariant operator on $\mathcal{H}_{\boldsymbol{\lambda}}\otimes\mathcal{S}_{L\mathfrak{p}}$, it is a constant on each of the irreducible subrepresentations of $L\mathfrak{h}$. If $\mathcal{U}_{\boldsymbol{\mu}}$ is the irreducible positive energy representation of $\tilde{L}\mathfrak{h}$ with lowest weight $\boldsymbol{\mu}$, then using (34), we see that the square of the Dirac operator takes the value

$$\begin{aligned}(\partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}})^2\big|_{\boldsymbol{\mu}} &= 2\,\varrho_{L\mathfrak{g}}(I)\,\varrho_{L\mathfrak{g}}(\partial_\theta) - \|\boldsymbol{\lambda} - \boldsymbol{\rho}_\mathfrak{g}\|^2\\ &\quad - 2\,\varrho'_{L\mathfrak{h}}(I)\,\varrho'_{L\mathfrak{h}}(\partial_\theta) + \|\boldsymbol{\mu} - \boldsymbol{\rho}_\mathfrak{h}\|^2 = -\|\boldsymbol{\lambda} - \boldsymbol{\rho}_\mathfrak{g}\|^2 + \|\boldsymbol{\mu} - \boldsymbol{\rho}_\mathfrak{h}\|^2,\end{aligned} \tag{36}$$

on the $\mathcal{U}_{\boldsymbol{\mu}}$ components of $\mathcal{H}_{\boldsymbol{\lambda}}\otimes\mathcal{S}_{L\mathfrak{p}}$. Note that the non-constant terms vanish since $\varrho_{L\mathfrak{g}}$ and $\varrho'_{L\mathfrak{h}}$ agree on $\mathbb{R}\,\tilde{\oplus}\,\tilde{L}\mathfrak{h}$.

Note that in the above construction, we are using the invariant inner product on $\mathfrak{h}$ obtained by restricting our invariant inner product on $\mathfrak{g}$. When $\mathfrak{g}$ is simple, we use the basic inner product on $\mathfrak{g}$, which is normalized so that $\|\alpha_{\max}\|^2 = 2$, where $\alpha_{\max}$ is the highest root of $\mathfrak{g}$. We recall that the basic inner product corresponds to the universal central extension $\tilde{L}\mathfrak{g}$ of $L\mathfrak{g}$, which in turn restricts to give a (not necessarily universal) central extension $\tilde{L}\mathfrak{h}$ of $L\mathfrak{h}$. Nevertheless, given any positive energy representation $\mathcal{H}$ of $\tilde{L}\mathfrak{g}$, the tensor product $\mathcal{H}\otimes\mathcal{S}_{L\mathfrak{g}}$ is a true representation of this central extension $\tilde{L}\mathfrak{h}$. So, if $\mathfrak{h}$ is reductive, then the squares of the Dirac operators $\partial\!\!\!/'_{L\mathfrak{h}}$ and $\partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}}$ are indeed of the form given by (34) and (36).

On the other hand, if $\mathfrak{g}$ is not simple but rather semi-simple, then the basic inner product on $\mathfrak{g}$ is normalized so that $\|\alpha_i\|^2 = 2$, where the α_i are the highest roots of each of the simple components of $\mathfrak{g}$. In this case, a projective positive energy representation of $L\mathfrak{g}$ is not necessarily a true representation of the corresponding central extension $\tilde{L}\mathfrak{g}$, so the expression (34) for the square of the Dirac operator on $L\mathfrak{g}$ is not universal. However, if $\mathfrak{h}$ is reductive, then the expression (36) for the square of the Dirac operator on $L\mathfrak{g}/L\mathfrak{h}$ does still hold, as the non-constant terms must vanish since the operator commutes with the action of $L\mathfrak{h}$.

10. The kernel of the Dirac operator

Given a linear operator $d : V \to W$ between two finite dimensional vector spaces, the alternating sum of the dimensions in the exact sequence

$$0 \longrightarrow \mathrm{Ker}\,d \longrightarrow V \xrightarrow{\;d\;} W \longrightarrow \mathrm{Coker}\,d \longrightarrow 0$$

vanishes, and it follows that $\operatorname{Index} d = \dim V - \dim W$. Furthermore, if V and W are G-modules and the operator d is G-equivariant, then the analogous result $\operatorname{Index}_G d = V - W$ holds in the representation ring $R(G)$. In the infinite dimensional case, this result does not necessarily hold, but for representations of loop groups, it does hold provided that the representations are of finite type and that the operator commutes with rotating the loops.

Lemma 13. *If $\mathcal{V}$ and $\mathcal{W}$ are representations of LG of finite type, and $\mathcal{D} : \mathcal{V} \to \mathcal{W}$ is an $S^1 \ltimes LG$-equivariant linear operator, then its LG-equivariant index is the virtual representation* $\operatorname{Index}_{LG} \mathcal{D} = \mathcal{V} - \mathcal{W}$.

Proof. Since $\mathcal{D}$ is S^1-equivariant, it respects the decompositions of $\mathcal{V}$ and $\mathcal{W}$ into their constant energy subspaces, and it can be written in the block diagonal form $\mathcal{D} = \bigoplus_{k\in\mathbb{Z}} \mathcal{D}_k$, with $\mathcal{D}_k : \mathcal{V}(k) \to \mathcal{W}(k)$. If both $\mathcal{V}$ and $\mathcal{W}$ are of finite type, then each of the subspaces $\mathcal{V}(k)$ and $\mathcal{W}(k)$ is a finite dimensional G-module, and so the $S^1 \times G$-equivariant index of $\mathcal{D}$ is given by the $R(G)$-valued formal power series

$$\operatorname{Index}_{S^1\times G} \mathcal{D} = \sum_{k\in\mathbb{Z}} z^k \big(\mathcal{V}(k) - \mathcal{W}(k)\big) = \sum_{k\in\mathbb{Z}} z^k\, \mathcal{V}(k) - \sum_{k\in\mathbb{Z}} z^k\, \mathcal{W}(k).$$

Since a representation of the full loop group LG is uniquely determined by its constant energy components, the LG-equivariant index must therefore be the difference of the domain and the range, hence $\operatorname{Index}_{LG} \mathcal{D} = \mathcal{V} - \mathcal{W}$. □

Returning to the notation of the previous section, let $\mathfrak{g}$ be semi-simple, and let $\mathfrak{h}$ be a reductive subalgebra of $\mathfrak{g}$ with maximal rank. If we decompose the spin representation as $\mathcal{S}_{L\mathfrak{p}} = \mathcal{S}^+_{L\mathfrak{p}} \oplus \mathcal{S}^-_{L\mathfrak{p}}$, the Dirac operator $\partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}}$ interchanges the two half-spin representations and can thus be written as the sum of the operators

$$\partial\!\!\!/^+_{L\mathfrak{g}/L\mathfrak{h}} : \mathcal{H}_{\boldsymbol{\lambda}} \otimes \mathcal{S}^+_{L\mathfrak{p}} \to \mathcal{H}_{\boldsymbol{\lambda}} \otimes \mathcal{S}^-_{L\mathfrak{p}},$$
$$\partial\!\!\!/^-_{L\mathfrak{g}/L\mathfrak{h}} : \mathcal{H}_{\boldsymbol{\lambda}} \otimes \mathcal{S}^-_{L\mathfrak{p}} \to \mathcal{H}_{\boldsymbol{\lambda}} \otimes \mathcal{S}^+_{L\mathfrak{p}},$$

where $\partial\!\!\!/^-_{L\mathfrak{g}/L\mathfrak{h}}$ is the adjoint of $\partial\!\!\!/^+_{L\mathfrak{g}/L\mathfrak{h}}$. When we introduced $\partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}}$ in (35), we showed that it is $L\mathfrak{h}$-equivariant, and all of our Dirac operators clearly commute with the generator ∂_θ of rotations of the loops. The operator $\partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}}$ is therefore $S^1 \ltimes LH$-equivariant, and since its domain and range are both of finite type, we may apply Lemma 13. The LH-equivariant index of $\partial\!\!\!/^+_{L\mathfrak{g}/L\mathfrak{h}}$ is thus the difference

$$\operatorname{Ker} \partial\!\!\!/^+_{L\mathfrak{g}/L\mathfrak{h}} - \operatorname{Ker} \partial\!\!\!/^-_{L\mathfrak{g}/L\mathfrak{h}} = \mathcal{H}_{\boldsymbol{\lambda}} \otimes \mathcal{S}^+_{L\mathfrak{p}} - \mathcal{H}_{\boldsymbol{\lambda}} \otimes \mathcal{S}^-_{L\mathfrak{p}},$$

which is given by the homogeneous Weyl-Kac formula (13).

On the other hand, to compute the kernel of $\partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}} = \partial\!\!\!/^+_{L\mathfrak{g}/L\mathfrak{h}} \oplus \partial\!\!\!/^-_{L\mathfrak{g}/L\mathfrak{h}}$ we proceed as in the computation of the kernel of the finite dimensional operator $\partial\!\!\!/_{\mathfrak{g}/\mathfrak{h}}$ in §6. In fact, the proofs of Lemmas 9 and 10 apply equally well in the Kac-Moody setting using the decomposition $\mathcal{S}_{L\mathfrak{g}/\mathfrak{t}} \cong \mathcal{S}_{L\mathfrak{p}} \otimes \mathcal{S}_{L\mathfrak{h}/\mathfrak{t}}$, and we obtain

Lemma 14. *For each $c \in \mathcal{C}$, the irreducible representation $\mathcal{U}_{c\bullet\boldsymbol{\lambda}}$ of $\tilde{L}\mathfrak{h}$ with lowest weight $c\bullet\boldsymbol{\lambda} = c(\boldsymbol{\lambda} - \boldsymbol{\rho}_\mathfrak{g}) + \boldsymbol{\rho}_\mathfrak{h}$ occurs exactly once in the decomposition of $\mathcal{H}_{\boldsymbol{\lambda}} \otimes \mathcal{S}_{L\mathfrak{p}}$.*

Lemma 15. *If $\boldsymbol{\mu}$ is a weight of $\mathcal{H}_{\boldsymbol{\lambda}} \otimes \mathcal{S}_{L\mathfrak{p}}$ satisfying $\|\boldsymbol{\mu} - \boldsymbol{\rho}_\mathfrak{h}\|^2 = \|\boldsymbol{\lambda} - \boldsymbol{\rho}_\mathfrak{g}\|^2$, then there exists a unique affine Weyl element $w \in \mathcal{W}_\mathfrak{g}$ such that $\boldsymbol{\mu} - \boldsymbol{\rho}_\mathfrak{h} = w(\boldsymbol{\lambda} - \boldsymbol{\rho}_\mathfrak{g})$.*

Then, in light of our formula (36) for the square of the Dirac operator $\partial\!\!\!/_{L\mathfrak{g}/L\mathfrak{h}}$, we immediately obtain the loop group analogue of Theorem 8.

Theorem 16. *Let $\mathfrak{g}$ be a semi-simple Lie algebra with a maximal rank reductive Lie subalgebra $\mathfrak{h}$. Let $\mathcal{H}_{\boldsymbol{\lambda}}$ and $\mathcal{U}_{\boldsymbol{\mu}}$ be the irreducible representations of $\tilde{L}\mathfrak{g}$ and $\tilde{L}\mathfrak{h}$ with lowest weights $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$. The kernel of the operator $\not{\partial}_{L\mathfrak{g}/L\mathfrak{h}}$ on $\mathcal{H}_{\boldsymbol{\lambda}} \otimes \mathcal{S}_{L\mathfrak{p}}$ is*

$$\operatorname{Ker} \not{\partial}_{L\mathfrak{g}/L\mathfrak{h}} = \bigoplus\nolimits_{c \in \mathcal{C}} \mathcal{U}_{c \bullet \boldsymbol{\lambda}},$$

where $c \bullet \boldsymbol{\lambda} = c(\boldsymbol{\lambda} - \boldsymbol{\rho}_{\mathfrak{g}}) + \boldsymbol{\rho}_{\mathfrak{h}}$, and $\mathcal{C} \subset \mathcal{W}_{\mathfrak{g}}$ is the subset of affine Weyl elements which map the fundamental Weyl alcove for $\mathfrak{g}$ into the fundamental alcove for $\mathfrak{h}$.

Comparing this result to the homogeneous Weyl-Kac formula (13), we obtain

$$\operatorname{Ker} \not{\partial}^{+}_{L\mathfrak{g}/L\mathfrak{h}} = \bigoplus_{(-1)^c = +1} \mathcal{U}_{c \bullet \boldsymbol{\lambda}}, \qquad \operatorname{Ker} \not{\partial}^{-}_{L\mathfrak{g}/L\mathfrak{h}} = \bigoplus_{(-1)^c = -1} \mathcal{U}_{c \bullet \boldsymbol{\lambda}}.$$

Taking the kernels of these Dirac operators therefore gives an explicit construction for the multiplet of signed representations of $\tilde{L}\mathfrak{h}$ corresponding to any given irreducible positive energy representation of $\tilde{L}\mathfrak{g}$.

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